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Control and Optimization Based on Network Communication

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Preface

With the rapid development of science and technology, networks play an important role in control systems. Typically, in a control system, the plant and the controller may communicate via a network. Such systems are called networked control systems and they have received significant attention in the literature. The introduction of a network has many advantages such as low cost, easy installation, and maintenance. However, a network also introduces some imperfections such as varying sampling, transmission delays, transmission protocols, packet losses, and so on. Besides, the network may also suffer from network attacks such as Denial of Service attacks. These imperfections may degrade the system performance and even make the closed-loop systems unstable. Therefore, it is necessary to investigate the effect of these imperfections on the stability and performance of control systems in a quantitative manner.

The content of this book consists of two parts. The first part aims at proposing modeling and analyzing approaches for network-based control systems under different problem settings. Model-based event-triggered transmission strategies are proposed to reduce the data transmissions in the network. We also consider the stability problem for networked control systems in the presence of stochastic noise and stochastic detecting instants. For all of these cases, hybrid models are established and Lyapunov functions are constructed. The second part first investigates the distributed unconstrained optimization problem which is solved over a network communication topology. The network may be destroyed by persistent attackers that result in communication failures. A distributed switching algorithm is proposed and a hybrid model is established to analyze stability. Then, the proposed method is extended to constrained aggregative games which are subject to unknown time-varying disturbances and unmodeled terms, and the communication topology is also influenced by attacks. This book provides a unifying framework to understand the complex dynamics of network-based control systems. The content is appropriate for students, control engineers, and scientists who are interested in network-based control and optimization problems.

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Notations

$\mathbb{R}^n$	n-dimensional Euclidean space
$\lvert\cdot\rvert$	2-norm of a matrix
$\mathbb{B}$	unit ball
$\mathbb{B}_r$	ball with radius r
$\langle x, y\rangle$	inner product of two vectors
x^T	transpose of vector x
$col(x,y)$	$\left(x^T, y^T\right)^T$
$\mathbb{Z}$	set of integers
$\mathbb{R}_{\geq 0}$	$[0, \infty)$
$\mathbb{R}_{\leq 0}$	$(-\infty, 0]$
$\mathbb{Z}_{\geq 0}$	set of non-negative integers
$\mathbb{Z}_{\leq 0}$	set of non-positive integers
$\mathbb{R}_{>0}$	$(0, \infty)$
$\mathbb{Z}_{>0}$	set of positive integers
$\mathbb{R}^N_{\geq 0}$	$\left\{ (x_1, \cdots, x_N)^T \middle\vert x_i \in \mathbb{R}_{\geq 0}\right\}$
$\mathbb{Z}^N_{\geq 0}$	$\left\{ (x_1, \cdots, x_N)^T \middle\vert x_i \in \mathbb{Z}_{\geq 0}\right\}$
$\lvert x\rvert_{\mathscr{W}}$	distance of a point x to a set $\mathscr{W}$
$\mathbf{0}_n$	n-dimensional zero matrix
I_n	n-dimensional identity matrix
$\mathbf{0}$	zero vector with appropriate dimension
$\mathbf{1}_n$	n-dimensional vector with all elements being 1
$diag\{M_1, \cdots, M_N\}$	block diagonal matrix
$\lambda_{max}(P)$	maximal eigenvalue of a positive definite matrix
$\lambda_{min}(P)$	minimal eigenvalue of a positive definite matrix
$A \otimes B$	Kronecker product of matrices A and B
$\overline{\text{con}}\Omega$	closed convex hull of a set
$\overline{\Omega}$	closure of a set
$x + \Omega$	$\{y \vert y = x + z, z \in \Omega\}$
$x \cdot \Omega$	$\{y \vert y = x \cdot z, z \in \Omega\}$
$int(\Omega)$	interior of a set Ω

$S+\varepsilon\mathbb{B}$	$\{x : \lvert x\rvert_S \leq \varepsilon\}$
$S+\varepsilon\mathbb{B}^{\circ}$	$\{x : \lvert x\rvert_S < \varepsilon\}$
S^p	$\left\{\left(x_1,\cdots,x_p\right)\lvert x_i \in S\right\}$
$sgn(x)$	$sgn(x)=1$, for $x>0$, 0, for $x=0$ and -1 for $x<0$
$sgn\left(\left(x_1^T,\cdots,x_n^T\right)^T\right)$	$(sgn(x_1), sgn(x_2),\cdots, sgn(x_n))^T$
$SGN(x)$	$SGN(x)=1$, for $x>0$, $[-1,1]$, for $x=0$ and -1 for $x<0$
$SGN\left(\left(x_1^T,\cdots,x_n^T\right)^T\right)$	$(SGN(x_1), SGN(x_2),\cdots, SGN(x_n))^T$

Chapter 1
Introduction

1.1 Network-Based Control

Networked control systems refer to control systems in which the plant and the controller communicate via a network channel. Introduction of the network brings some benefits such as low cost, easy installation and maintenance. On the other hand, the network also introduces some imperfections such as varying transmission intervals, varying transmission delays, communication constraints, quantization effects and packet losses. The network may also suffer from various kinds of attacks such as denial-of-service attacks which can block the transmission channels. Therefore, it is necessary to investigate the effects of various kinds of imperfections on stability and performance of the closed-loop systems in a quantitative manner. Besides, in networked control systems, another important problem is how to reduce the data transmissions because the network resources are limited. There are mainly two transmission strategies. One is the time-triggered transmission strategy, which triggers the data transmission periodically, and the other is the event-triggered transmission strategy, which triggers the data transmission based on a triggering condition involving the system state and error variable. The latter has the potential to reduce data transmissions because a transmission happens only when it is necessary.

In networked control systems, there are mainly two modeling approaches. One is the delay system approach, which models the sampled-data control system into an input delay system and then the Lyapunov functional approach is applied to analyze the stability and performance of the resulted closed-loop system. The delay system approach has the advantage of less conservativeness but is often restricted to linear systems. The other modeling approach is the hybrid system approach, which models the networked control system as a hybrid system. The hybrid system approach is more appropriate for nonlinear systems and is flexible enough to model various kinds of complex dynamical systems such as event-triggered control systems.

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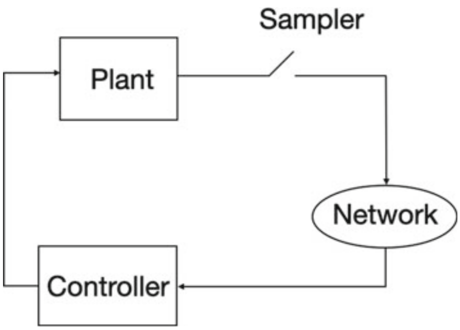

Fig. 1.1 Structure of networked control systems

Figure 1.1 illustrates a simple structure of networked control systems. The dynamics of the plant can be described by the following differential equation

$$\dot{x} = f(x, u). \tag{1.1}$$

A network-based controller can be used to stabilize the plant and its mathematical form is given as follows

$$u(t) = k(x(t_i)), \quad t \in [t_i, t_{i+1}). \tag{1.2}$$

where t_i is a sequence of transmission instants. Define an error variable $e(t) := x(t_i) - x(t), t \in [t_i, t_{i+1})$. Then the error variable e will have jump dynamics at the transmission instants. Specifically, once a transmission happens and the controller has received the transmitted data, e will be reset as zero. Therefore, e has continuous dynamics and discrete jump dynamics. Specifically, the continuous dynamics of x and e can be described by the following differential equation

$$\begin{aligned}\dot{x} &= f(x, k(x + e)) \\ \dot{e} &= -f(x, k(x + e)).\end{aligned}$$

The discrete dynamics of x and e can be described by the following difference equation

$$\begin{aligned}x^+ &= x \\ e^+ &= 0.\end{aligned}$$

Actually, real networked control systems are very complex due to the effect of various kinds of network factors. The transmitted data can arrive at the destination only after some time of delay. Networked control systems may contain many nodes and the sensors in each node collect part of the system data. Due to the limitation of communication bandwidth, only part of the sampled data can be transmitted via the network to the destination. Transmission protocols are therefore introduced to

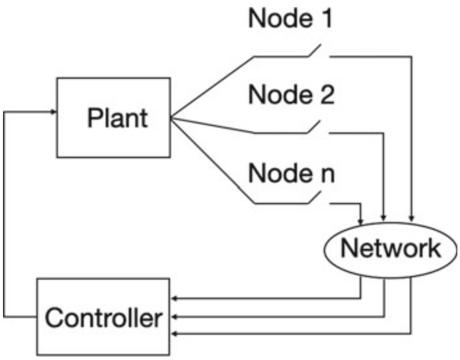

Fig. 1.2 Structure of networked control systems with multiple nodes

schedule the transmission order of the data in different nodes. The most widely known transmission protocols are the Round-Robin protocol, Try-Once-Discard protocol and stochastic protocols. For the Round Robin protocol, each node is authorized to have access to the network periodically. For the Try-Once-Discard protocol, all the nodes compete with each other and the node with the maximal error weight has the access to the network. For the stochastic transmission protocols, the nodes are authorized by stochastic scheduling rules. Figure 1.2 illustrates the structure of networked control systems containing multiple nodes. We will use the hybrid system framework to model networked control systems under different problem settings and provide corresponding analyzing approaches.

1.2 Basic Knowledge About Hybrid Systems

A hybrid dynamical system refers to dynamical systems that exhibit continuous-time dynamics and discrete-time dynamics [22, 31, 44, 76]. In this book, we use the hybrid framework proposed in [22] for our considered problems.

1.2.1 Hybrid Dynamical Systems

We introduce some basic knowledge about hybrid dynamical systems [22] characterized by

$$\begin{cases} \dot{x} \in F(x), \ x \in C \\ x^+ \in G(x), \ x \in D, \end{cases} \tag{1.3}$$

where $C \subset \mathbb{R}^n$ is the *flow set*, $D \subset \mathbb{R}^n$ is the *jump set*, the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *flow map* and the set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *jump* *map*. System (1.3) is represented by the notation $\mathscr{H} := \{C, F, D, G\}$, where C, F, D and G comprise the data of $\mathscr{H}$. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1}([s_j, s_{j+1}], j)$ for some finite sequence of times $0 = s_0 \leq s_1 \leq s_2 \leq \ldots s_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid domain.

Definition 1.1 [22] A function $x : dom\ x \mapsto \mathbb{R}^n$ is a *hybrid arc* if $dom\ x$ is a hybrid time domain and $t \mapsto x(t, j)$ is locally absolutely continuous for each j such that the interval $I_j := \{t : (t, j) \in dom\ x\}$ has nonempty interior. A hybrid arc is complete if its domain is unbounded. A hybrid arc x is a solution to system (1.3) if $x(0, 0) \in \overline{C} \cup D$, and the following two conditions hold:

(1) for all $j \in \mathbb{Z}_{\geq 0}$ such that I_j has nonempty interior

$$\begin{aligned} &x(t, j) \in C \quad for\ all\ t \in int(I_j), \\ &\dot{x}(t, j) \in F(x(t, j)) \quad for\ almost\ all\ t \in I_j; \end{aligned}$$

(2) for all $(t, j) \in dom\ x$ such that $(t, j+1) \in dom\ x$,

$$\begin{aligned} x(t, j) &\in D, \\ x(t, j+1) &\in G(x(t, j)). \end{aligned}$$

The following two definitions characterize two special classes of functions that will be often used.

Definition 1.2 A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a class-$\mathscr{K}$ function if it is continuous, zero at zero and strictly increasing. $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathscr{K}_\infty$ function if it is a class-$\mathscr{K}$ function and unbounded. It is a class-$\mathscr{G}_\infty$ function, if it is continuous, nondecreasing and unbounded.

Definition 1.3 A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a class $\mathscr{K}\mathscr{L}$ function if it is nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s \to 0^-} \beta(s, t) = \lim_{t \to \infty} \beta(s, t) = 0$.

The following definition characterizes a class of set-valued mappings owning some continuity properties.

Definition 1.4 A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semi-continuous (OSC) at $x \in \mathbb{R}^n$ if for all sequences $x_i \to x$ and $y_i \in M(x_i)$ such that $y_i \to y$ we have that $y \in M(x)$. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is locally bounded (LB) at $x \in \mathbb{R}^n$ if there exists a neighborhood U_x of x such that $M(U_x) \subset \mathbb{R}^m$ is bounded. Given a set $\Omega \subset \mathbb{R}^n$, the mapping M is said to be OSC and LB relative to Ω if the set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $M(x)$ for $x \in \Omega$ and $\emptyset$ for $x \notin \Omega$ is OSC and LB at each $x \in \Omega$.

We next give stability definitions for hybrid systems.

Definition 1.5 [22] Consider a hybrid system $\mathscr{H}$ on $\mathbb{R}^n$. Let $\mathscr{W} \subset \mathbb{R}^n$ be closed. The set $\mathscr{W}$ is said to be

- uniformly asymptotically stable if there exist a class-$\mathscr{KL}$ function β and a positive constant c, such that for any solution x to $\mathscr{H}$ with $|x(0,0)|_{\mathscr{W}} < c$,

$$|x(t,j)|_{\mathscr{W}} \leq \beta(|x(0,0)|_{\mathscr{W}}, t+j),\ \forall (t,j) \in dom\ x; \tag{1.4}$$

- uniformly globally asymptotically stable if inequality (1.4) is satisfied for any solution to $\mathscr{H}$.

1.2.2 Stochastic Hybrid Systems

Consider the following stochastic hybrid systems proposed in [83]

$$\begin{cases} dx = F(x)dt + B(x)dw \\ x^+ = G(x,v) \end{cases} \tag{1.5}$$

where $x \in \mathbb{R}^n$ is the state variable and w is a brownian motion. $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ are respectively the flow set and jump set. (F, B) and G are respectively the flow map and jump map. v is a placeholder for independent, identically distributed (i.i.d.) random variables. F and B are continuous functions. $x \mapsto G(x,v)$ is a continuous function for each $v \in \mathbb{R}^q$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a given probability space. Let $\{\mathrm{v}_i\}_{i=1}^{\infty}$ with $\mathrm{v}_i : \Omega \to \mathbb{R}^q$ be the sequence of i.i.d. random variables, defined on this probability space. Let $\hat{\mathscr{F}}_0 := \{\Omega, \emptyset\}$, and let $\{\hat{\mathscr{F}}_i\}_{i=1}^{\infty}$ denote the minimal filtration associated to the random process $\{\mathrm{v}_i\}_{i=1}^{\infty}$; that is, $\hat{\mathscr{F}}_i$ is the σ-algebra generated by $(\mathrm{v}_1, \cdots, \mathrm{v}_i)$. Define $\hat{\mathbb{F}} := \{\hat{\mathscr{F}}_j\}_{j\in\mathbb{Z}_{\geq 0}}$. The additional filtration $\tilde{\mathbb{F}} := \{\tilde{\mathscr{F}}_t\}_{t\in\mathbb{R}_{\geq 0}}$ of $(\Omega, \mathscr{F}, \mathbb{P})$ is independent of $\hat{\mathbb{F}}$ and is such that $\tilde{\mathscr{F}}_0$ contains all $\mathbb{P}$-negligible subsets in $\mathscr{F}$.

Definition 1.6 [83] Given the filtered probability space $(\Omega, \mathscr{F}, \tilde{\mathbb{F}}, \mathbb{P})$, an $\tilde{\mathbb{F}}$-adapted Brownian motion relative to this space is a continuous process $\mathrm{w} = \{\mathrm{w}(t,\cdot)\}_{t\geq 0}$ with the properties that $\sigma(\mathrm{w}(s,\cdot))$, $s \in [0,t] \subset \tilde{\mathscr{F}}_t$ for all $t \geq 0$, $\mathrm{w}(0,\cdot) = 0$ almost surely and for $0 \leq s < t$ the increment $\mathrm{w}(t,\cdot) - \mathrm{w}(s,\cdot)$ is normally distributed with mean zero and variance $t-s$ and is independent of $\tilde{\mathscr{F}}_s$.

Define the hybrid filtration [83]

$$\mathbb{F} := \{\mathscr{F}_{t,j} := \bigcap_{\epsilon > 0} \sigma(\tilde{\mathscr{F}}_{t+\epsilon}, \hat{\mathscr{F}}_j)\}_{(t,j)\in\mathbb{R}_{\geq 0}\times\mathbb{Z}_{\geq 0}}. \tag{1.6}$$

Definition 1.7 [83] A relaxed hybrid arc is a mapping $\phi : H \to \mathbb{R}^n$ such that H is a hybrid time domain and, for each $j \in \mathbb{Z}_{\geq 0}$, $t \mapsto \phi(t,j)$ is continuous. A stochastic hybrid arc is a mapping x defined on Ω such that $\mathrm{x}(\omega)$ is a relaxed hybrid arc for each $\omega \in \Omega$ and the set-valued mapping from Ω to $\mathbb{R}^{n+2}$ defined by

$$\omega \mapsto \mathrm{graph}(\mathrm{x}(\omega)) := \{(t, j, z) \in \mathbb{R}^{n+2} : (t, j) \in \mathrm{dom}\mathrm{x}_\omega,\ \ z = \mathrm{x}_\omega(t, j)\}$$

is $\mathscr{F}$-measurable with closed values. An $\mathbb{F}$-adapted stochastic hybrid arc is a stochastic hybrid arc x such that the set-valued mapping

$$\omega \mapsto \mathrm{graph}(\mathrm{x}(\omega)) \cap ([0, t] \times \{0, \cdots, j\} \times \mathbb{R}^n)$$

is $\mathscr{F}_{t,j}$-measurable for each $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

Definition 1.8 [83] Given the sequence of i.i.d. random variables $\mathrm{v} := \{\mathrm{v}_i\}_{i=1}^{\infty}$, a solution starting at $x \in \mathbb{R}^n$ is a pair (x, w), where

- x is an $\mathbb{F}$-adapted stochastic hybrid arc,
- w is an $\tilde{\mathbb{F}}$-adapted Brownian motion

such that $\mathrm{x}(\omega)$ is a solution starting at x with the inputs $\mathrm{w}(\cdot, \omega)$, $\mathrm{v}(\omega)$ for every ω. That is, for almost every $\omega \in \Omega$,

- $\mathrm{x}_\omega(0, 0) = x$;
- for each $j \in \mathbb{Z}_{\geq 0}$, if $I^j(\omega) := \{t : (t, j) \in \mathrm{x}(\omega)\}$ has nonempty interior then, for every $t \in I^j(\omega)$,
  (a) $\mathrm{x}_\omega(t, j) \in C$
  (b) $\mathrm{x}_\omega(t, j) - \mathrm{x}_\omega(T_j, j) = \int_{T_j}^{t} F ds + \int_{T_j}^{t} B d\mathrm{w}(s, w)$
- if $(t, j), (t, j + 1) \in \mathrm{dom}\ \mathrm{x}(\omega)$ then
  (a) $\mathrm{x}_\omega(t, j) \in D$
  (b) $\mathrm{x}_\omega(t, j + 1) = G(\mathrm{x}_\omega(t, j), \mathrm{v}_{j+1}(\omega))$.

For a solution (x, w) starting at a point in K, we use the shorthand notation $\mathrm{x} \in \mathscr{S}_r(K)$.

Next two important stability concepts are introduced for stochastic hybrid systems.

Definition 1.9 [83] A compact set $\mathscr{A}$ is said to be

- *uniformly Lyapunov stable in probability* if for each $\epsilon > 0$ and $\rho > 0$ there exists $\delta > 0$ such that

$$\mathrm{x} \in \mathscr{S}_r(\mathscr{A} + \delta\mathbb{B}) \Rightarrow \mathbb{P}(\mathrm{graph}(\mathrm{x}) \subset \mathbb{R}^2 \times (\mathscr{A} + \epsilon\mathbb{B})) \geq 1 - \rho, \tag{1.7}$$

- *uniformly Lagrange stable in probability* if for each $\delta > 0$ and $\rho > 0$ there exists $\epsilon > 0$ such that (1.7) holds,
- *uniformly globally stable in probability* if it is both uniformly Lyapunov stable in probability and uniformly Lagrange stable in probability,
- *uniformly globally attractive in probability* if for each $\Delta > 0$, $\epsilon > 0$ and $\rho > 0$ there exists $\tau > 0$ such that

$$\mathrm{x} \in \mathscr{S}_r(\mathscr{A} + \Delta\mathbb{B}) \Rightarrow \mathbb{P}(\mathrm{graph}(\mathrm{x}) \cap (\Gamma_{\geq\tau} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times (\mathscr{A} + \epsilon\mathbb{B}^\circ)) \geq 1 - \rho$$

where $\Gamma_{\geq\tau} := \{(s, i) : (s, i) \in \mathbb{R}^2,\ s + i \geq \tau\}$,

- *uniformly globally asymptotically stable in probability* (UGASp) if it is uniformly globally stable in probability and uniformly globally attractive in probability.

Part I
Modeling and Stability Analysis for Networked Control Systems

Chapter 2
Model-Based Event-Triggered Control for Distributed Networked Control Systems

2.1 Introduction

Event-triggered sampled-data control has received a lot of attention in recent years due to its potential advantage over time-triggered sampled-data control, namely that it may reduce the data transmission in the network [12, 19, 21, 40, 51, 54, 66, 73, 80, 87, 96, 104, 106, 107, 111]. Most of existing literature on event-triggered transmission strategies use zero-order-hold between two updating times on the controller side once the feedback data is received (see, for example, [80, 107] and references cited therein). The zero-order-hold has the advantage to be implemented easily in sampled-data control systems while it may result in large error between plant state and data kept by the zero-order-hold at the controller side. Differently, model-based sampled-data control sufficiently utilizes the model information at the controller side and, between two updating times, the controller predicts the plant state based on the models and received feedback data. With the model-based controller, less data transmission may be expected by the controller designers [27, 59, 105]. Considering model-based control and event-triggered sampled-data control at the same time is thus an appealing research topic.

The author in [80] proposes an event-triggered scheduling rule based on an inequality relation involving the plant state and state kept by zero-order-hold at the controller side, and proves that Zeno behavior will not happen under the designed transmission condition. Different from [80], the author in [21] proposes a dynamic event-triggered transmission strategy which needs to detect whether the state value of an auxiliary integrator with positive initial state arrives at zero. Guaranteeing asymptotic stability, the authors in [2, 73] consider the event-triggered control for systems with output feedback and impose a waiting time between two transmission times to avoid Zeno behavior. In [13], the authors consider small delays and transmission protocols in each local network for interconnected nonlinear systems with output feedback, and propose static and dynamic event-triggered transmission strategy by imposing a waiting time in each local network.

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The previously mentioned literature uses a zero-order-hold at the controller side which maintains the received feedback data between two updating times. In [18, 27], the authors consider the model-based event-triggered transmission strategy. The authors in [18] consider transmission delays and quantization effects in the network for uncertain linear systems and the proposed transmission strategy is static. Zeno behavior is precluded by proving that there is a lower bound between the inter-execution times. The authors in [27] consider linear discrete-time systems with disturbances and propose a periodic event-triggered transmission strategy. Moreover, for decentralized linear systems, the authors in [27] use a reduced model of each subsystem to predict the subsystem state for the packet dropout situation. Differently, this chapter aims at designing centralized and distributed model-based dynamic event-triggered transmission strategies. The literature closest to our work is [13, 18, 27]; while [18] considers static and centralized event-triggered transmission strategies, [27] considers discrete-time systems without delays and protocols and [13] uses a zero-order-hold.

The main content of this chapter can be summarized as follows. A model-based dynamic event-triggered transmission strategy is proposed for uncertain sampled-data linear systems with transmission delays and protocols in the network. By introducing storage variables, the entire systems are modeled as hybrid systems and then based on stability theorems of hybrid systems, the explicit parameters of dynamic equation involving the transmission strategy are designed such that the hybrid systems are asymptotically stable. Moreover, for uncertain linear systems that can be decomposed into interconnected subsystems, a distributed model-based dynamic event-triggered transmission strategy is also proposed. In each local network, the local controller runs a reduced model that does not use information about other subsystems. Again based on a hybrid system model, the parameter design is explicitly presented and asymptotic stability is guaranteed.

2.2 Model Description

Consider the following plant

$$\dot{x} = Ax(t) + Bu(t) \tag{2.1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, and $u \in \mathbb{R}^{n_u}$ is the control input. The following model-based controller is used to stabilize system (2.1):

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x}(t) + \hat{B}u(t) \\ u(t) &= K\hat{x}(t) \end{aligned} \tag{2.2}$$

where $\hat{x} \in \mathbb{R}^{n_x}$ is the state of the model, $\hat{A}$ and $\hat{B}$ are the ideal parameters of system matrices. We will denote $A_0 := \hat{A} + \hat{B}K$, $B_0 := \hat{B}K$, $\varDelta A_0 := A + BK - \hat{A} - \hat{B}K$

and $\Delta B_0 := BK - \hat{B}K$ for brevity. We consider the case that the state of system (2.1) is sampled by the sensors and transmitted via the network at times $t_i \geq 0, i \in \mathbb{Z}_{\geq 0}$ to the controller. It is assumed that there are ℓ nodes in the network and the state x is separated into $x = (x_1, x_2, \ldots, x_\ell)$ correspondingly with $x_j \in \mathbb{R}^{n_{x_j}}$. At each transmission time t_i, only one node $j(t_i) \in \{1, 2, \ldots, \ell\}$ is authorized by the transmission protocols to transmit the data $x_{j(t_i)}$ to the controller.

The transmitted data $x_j(t_i)$, $j \in \{1, 2, \ldots, \ell\}$ will arrive at the controller at $t_i + d_i, d_i \geq 0$ after certain transmission delay $d_i < t_{i+1} - t_i$. Once the controller receives the transmitted data at $t_i + d_i$, $\hat{x}$ will update based on the following equation

$$\hat{x}((t_i + d_i)^+) = e^{A_0 d_i} \left((I_{n_x} - \Phi(i))\hat{x}(t_i) + \Phi(i)x(t_i)\right) \tag{2.3}$$

where $\Phi(i)$ is a block diagonal matrix defined as

$$\Phi(i) := diag(\mathbf{0}_{n_{x_1}}, \ldots, \mathbf{0}_{n_{x_{j-1}}}, I_{n_{x_j}}, \mathbf{0}_{n_{x_{j+1}}}, \ldots, \mathbf{0}_{n_{x_\ell}})$$

and j in $\Phi(i)$ means that the node j is authorized by the transmission protocol at time t_i.

Update of $\hat{x}$ at $t_i + d_i$ based on (2.3) can be realized through the time stamp technique in the network. Once the controller receives a packet including the information $x_j(t_i)$ and corresponding time stamp, the controller will compute the transmission delay d_i and $\hat{x}((t_i + d_i)^+)$ by combining the past information $\hat{x}(t_i)$ stored in the memory of the controller. The reason that $\hat{x}$ updates in the form (2.3) is that the sensors also run a model that is same as that in the controller

$$\dot{\bar{x}} = A_0\bar{x}, \quad t \in [t_i, t_{i+1}]$$

which updates at t_i according to

$$\bar{x}(t_i^+) = (I_{n_x} - \Phi(i))\bar{x}(t_i) + \Phi(i)x(t_i).$$

Under the same initial condition $\bar{x}(t_0) = \hat{x}(t_0)$, the updating Eq. (2.3) will ensure $\hat{x}((t_i + d_i)^+) = \bar{x}((t_i + d_i)^+)$ and thus $\hat{x}(t) = \bar{x}(t)$, $t \in (t_i + d_i, t_{i+1}]$ which is used to ensure stability of closed-loop system if we realize that the transmission protocols, for example, Try-Once-Discard protocols [64] and the event-triggered transmission strategy introduced below can access only the plant state and model state in the sensors. The updating Eq. (2.3) can also be found in [18], which considers only one node in the network.

Denote $e := \hat{x} - x \in \mathbb{R}^{n_e}$ with $n_e = n_x$. Then from (2.3), we have

$$\hat{x}((t_i + d_i)^+) = e^{A_0 d_i} \left(h(i, e(t_i)) + x(t_i)\right) \tag{2.4}$$

where for brevity, we denote $h(i, e) := (I_{n_x} - \Phi(i))e$ and call $h(i, e)$ the transmission protocol.

From (2.4), it can be seen that the jumping dynamics at $t_i + d_i$ depend on the past state. Therefore, we introduce a storage variable $\hat{s}$ which has continuous dynamics

$$\dot{\hat{s}} = \begin{cases} A_0 \hat{s}(t), & t \in [t_i, t_i + d_i] \\ \dot{x}, & t \in [t_i + d_i, t_{i+1}] \end{cases} \tag{2.5}$$

and discrete jump dynamics

$$\hat{s}(t_i^+) = h(i, e(t_i)) + x(t_i), \quad \hat{s}((t_i + d_i)^+) = x(t_i + d_i). \tag{2.6}$$

From (2.4), (2.5) and (2.6), we can see that

$$\begin{aligned} \hat{x}((t_i + d_i)^+) &= \hat{s}(t_i + d_i) \\ &= e^{A_0 d_i}(h(i, e(t_i)) + x(t_i)). \end{aligned}$$

Replace variable $\hat{s}$ with $\hat{s} = s + x$. Then s will be subject to the continuous dynamics

$$\dot{s} := \begin{cases} A_0 s(t) - \varDelta A_0 x - BKe, & t \in [t_i, t_i + d_i] \\ 0, & t \in [t_i + d_i, t_{i+1}] \end{cases} \tag{2.7}$$

and discrete jump dynamics as follows

$$s(t_i^+) = h(i, e(t_i)), \quad s((t_i + d_i)^+) = 0. \tag{2.8}$$

As a result, the jump dynamics of e can be represented as

$$\begin{aligned} e((t_i + d_i)^+) &= \hat{x}((t_i + d_i)^+) - x(t_i + d_i) \\ &= \hat{s}(t_i + d_i) - x(t_i + d_i) \\ &= s(t_i + d_i). \end{aligned} \tag{2.9}$$

The sampling sequence $\{t_i\}_{i=0}^{\infty}$ is assumed to be generated by the following event-triggered transmission strategy

$$t_{i+1} := \inf\{t | t \geq t_i + T, \upsilon(t) \leq 0\} \tag{2.10}$$

where $T > 0$ is a waiting time used to avoid the Zeno behavior and υ is subject to the following dynamic equation which will be designed with initial value $\upsilon(t_0) > 0$:

$$\dot{\upsilon} = \begin{cases} f_{\upsilon,0}(\upsilon, x(t)), & t \in [t_i, t_i + T] \\ f_{\upsilon,1}(\upsilon, x(t), e(t)), & t \in [t_i + T, t_{i+1}]. \end{cases} \tag{2.11}$$

The transmission strategy (2.10) is a dynamic event-triggered transmission strategy since its triggering condition depends on the solution of the dynamic equation. The following assumptions are used.

Assumption 2.1 The transmission delays d_i satisfy $0 \leq d_i \leq \tau^* \leq T$. □

Assumption 2.2 $\delta_{A_0} > 0$ and $\delta_{B_0} > 0$ are such that the model uncertainties satisfy $|\Delta A_0| \leq \delta_{A_0}$ and $|\Delta B_0| \leq \delta_{B_0}$. □

Assumption 2.3 [29] The transmission protocol $h(\kappa, e)$ is uniformly globally exponentially stable (UGES) which means that there exist a continuous function $W : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_e} \to \mathbb{R}$ which is locally Lipschitz continuous in its second variable, constants $a_i > 0 (i = 1, 2)$ and $\lambda \in (0, 1)$ such that for all $\kappa \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{R}^{n_e}$, the following conditions hold:

$$
\begin{aligned}
a_1|e| \leq W(\kappa, e) &\leq a_2|e| \\
W(\kappa + 1, h(\kappa, e)) &\leq \lambda W(\kappa, e).
\end{aligned}
$$

Moreover, there exist constants $M > 0$, $\lambda_W \geq 1$ such that for all $\kappa \in \mathbb{Z}_{\geq 0}$ and almost all $e \in \mathbb{R}^{n_e}$, it holds that

$$\left|\frac{\partial W(\kappa, e)}{\partial e}\right| \leq M$$

and for all $\kappa \in \mathbb{Z}_{\geq 0}$ and all $e \in \mathbb{R}^{n_e}$, we have such that

$$W(\kappa + 1, e) \leq \lambda_W W(\kappa, e).$$ □

Combining (2.1), (2.2), (2.7), (2.8), (2.9), (2.10) and (2.11), we can model the resulting networked control systems into a hybrid system [22] with flow dynamics consisting of three subsystems in three subsets of the flow set

$$
\begin{cases}
\dot{z} = f(z), & z \in C \\
z^+ = g(z), & z \in D
\end{cases}
\tag{2.12}
$$

where $z := (x, e, s, \upsilon, l, \tau, \kappa)$ is the state variable, $C := C_1 \cup C_2 \cup C_3$, $D := D_1 \cup D_2 \cup D_3$ which are specified in the following description. The flow dynamics are governed by

$$z \in C \qquad \begin{cases} \dot{x} = f_1(x, e) \\ \dot{e} = f_2(x, e) \\ \dot{s} = f_3(x, e, s) \\ \dot{\upsilon} = f_\upsilon(x, e, \upsilon) \\ \dot{l} = 0 \\ \dot{\tau} = 1 \\ \dot{\kappa} = 0 \end{cases} \tag{2.13}$$

where

$$\begin{aligned} f_1(x, e) &:= (A_0 + \varDelta A_0)x + (B_0 + \varDelta B_0)e \\ f_2(x, e) &:= -\varDelta A_0 x + (\hat{A} - \varDelta B_0)e \\ f_3(x, e, s) &:= \begin{cases} A_0 s - \varDelta A_0 x - BKe, & z \in C_1 \\ 0, & z \in C_2 \cup C_3 \end{cases} \\ f_\upsilon(x, e, \upsilon) &:= \begin{cases} f_{\upsilon,0}(\upsilon, x), & z \in C_1 \cup C_2 \\ f_{\upsilon,1}(\upsilon, x, e), & z \in C_3 \end{cases} \end{aligned}$$

and

$$\begin{aligned} C_1 &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_s} \times \mathbb{R}_{\geq 0} \times \{1\} \times [0, \tau^*] \times \mathbb{Z}_{\geq 0}, \\ C_2 &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \{\mathbf{0}\} \times \mathbb{R}_{\geq 0} \times \{0\} \times [0, T] \times \mathbb{Z}_{\geq 0}, \\ C_3 &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \{\mathbf{0}\} \times \mathbb{R}_{\geq 0} \times \{2\} \times [T, \infty) \times \mathbb{Z}_{\geq 0}. \end{aligned}$$

τ is a timer variable, κ is a counter variable that keeps track of the sampling times and l is a logic variable used to identify whether a transmitted data arrives at the controller.

The jump dynamics are subject to the following equations

$$\begin{aligned} G(x, e, s, \upsilon, l, \tau, \kappa) &= (x, s, 0, \upsilon, 0, \tau, \kappa), & z \in D_1 \\ G(x, e, s, \upsilon, l, \tau, \kappa) &= (x, e, s, \upsilon, 2, \tau, \kappa), & z \in D_2 \\ G(x, e, s, \upsilon, l, \tau, \kappa) &= (x, e, h(\kappa, e), \upsilon, 0, 0, \kappa + 1), & z \in D_3 \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} D_1 &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_s} \times \mathbb{R}_{\geq 0} \times \{1\} \times [0, \tau^*] \times \mathbb{Z}_{\geq 0}, \\ D_2 &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \{\mathbf{0}\} \times \mathbb{R}_{\geq 0} \times \{0\} \times \{T\} \times \mathbb{Z}_{\geq 0} \\ D_3 &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \{\mathbf{0}\} \times \{0\} \times \{2\} \times [T, \infty) \times \mathbb{Z}_{\geq 0}. \end{aligned}$$

The flow dynamics (2.13) characterize the situation that the sampled data still does not arrive at the controller and also characterize the situation that the sampled data already arrives at the controller while the value of the timer stills does not exceed the waiting time. Moreover, the flow dynamics (2.13) also characterize the case that the value of the timer exceeds the waiting time while the transmission condition is still not satisfied. Introducing a storage variable s in the model is inspired by the paper [29], which considers the zero-order-hold case while we consider a model-based controller. In [29], the storage variable is subject to $\dot{s} = 0$ in the flow set.

From (2.10), the transmission is triggered immediately once υ arrives at zero, while from the flow set in systems (2.12), a solution of systems (2.12) may still evolve continuously when υ arrives at zero. The hybrid systems (2.12) can generate a set of solutions which is larger than that generated by (2.10). Therefore, stability of the closed-loop networked control systems under event-triggered transmission strategy (2.10) can be deduced by stability of hybrid systems (2.12).

2.3 Design of Triggering Function and Stability Analysis

In Sect. 2.2, we modeled the event-triggered control systems as a hybrid system. In this section, we will give explicit designs of the triggering conditions such that the closed-loop systems are stable.

Define the following functions $W_l : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_s} \to \mathbb{R}_{\geq 0}$:

$$\begin{aligned} W_0(\kappa, e, s) &:= W(\kappa, e) \\ W_1(\kappa, e, s) &:= \max\{W(\kappa, s), \frac{\lambda}{\lambda_W} W(\kappa, e)\}, \\ W_2(\kappa, e, s) &:= W(\kappa, e) \end{aligned}$$

where W satisfies the conditions in Assumption 2.3. From the definitions of W_0, W_1, W_2, it is clear to see that there exist $b_{l1}, b_{l2} > 0$ such that $b_{l1}|(e, s)|^2 \leq W_l^2(\kappa, e, s) \leq b_{l2}|(e, s)|^2$ for all $z \in C \cup D$ with z specified below (2.12) by combining Assumption 2.3 and triangle inequality of vector norm. These two functions will be used as the energy functions of (e, s) subsystems in different subsets.

Using Assumption 2.3, we can conclude the following lemma

Lemma 2.1 *Suppose Assumption 2.3 holds. Then for all $\kappa \in \mathbb{Z}_{\geq 0}, x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}, s \in \mathbb{R}^{n_s}$, it holds that*

$$\left| \frac{\partial W_l(\kappa, e, s)}{\partial (e, s)} (f_2(x, e), f_3(x, e, s)) \right| \leq L_l W_l + H_l(x)$$

where $L_0 = L_2 := \frac{M(|\hat{A}|+\delta_{B_0})}{a_1}$, $H_l(x) := M\delta_{A_0}|x|$, and

$$L_1 := \max\left\{ \frac{M(|\hat{A}| + \delta_{B_0})}{a_1}, \frac{M|A_0|\lambda + M|BK|\lambda_W}{a_1 \lambda} \right\}. \qquad \square$$

We are ready to present one of the main conditions in this chapter before the main result is presented.

Condition 2.1 *There exist constants* $\alpha_1, \alpha_2 > 0$, *positive definite functions* $\phi, \varphi : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ *and* $\chi : \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$, *constants* $\delta > 0, \gamma_1 > 0, \gamma_0 = \gamma_2 > 0$, *and a continuously differentiable function* $V : \mathbb{R}^{n_x} \to \mathbb{R}$ *such that the following conditions hold*

(C1) $\alpha_1 |x|^2 \leq V(x) \leq \alpha_2 |x|^2$ *for all* $x \in \mathbb{R}^{n_x}$,

(C2) for all $z \in C$

$$\frac{\partial V}{\partial x} f_1 \leq -\delta |(x, e, s)|^2 - \phi(x) - H_\ell^2(x) + \gamma_\ell^2 W_\ell^2(\kappa, e, s),$$

(C3) for all $\kappa \in \mathbb{Z}_{\geq 0}$, $x \in \mathbb{R}^{n_x}$ *and* $e \in \mathbb{R}^{n_e}$,

$$H_0^2(x) + \chi(e) + \phi(x) \geq \gamma_0^2 W^2(\kappa, e) + 2\lambda\gamma_0 W(\kappa, e)(L_0 W(\kappa, e) + H_0(x)) + \varphi(x).$$

□

Remark 2.1 (C1) of Condition 2.1 is trivial. (C2) are essentially input-to-state stability conditions. Roughly speaking, (C2) means that different gains are assigned to subsystems of (e, s). Since the system is linear, the continuously differentiable function V is actually chosen as $V(x) := x^T P x$. If A_0 is Hurwitz and the uncertainties are small then Condition 2.1 can always be satisfied with appropriate parameters.

In hybrid systems (2.12), the parameters T and τ^* are also needed to be designed. To solve for these two parameters, consider the following two ordinary differential equations firstly motivated by [29]

$$\begin{aligned} \dot{\pi}_0 &= -2L_0\pi_0 - \gamma_0\pi_0^2 - \gamma_0 \\ \dot{\pi}_1 &= -2L_1\pi_1 - \gamma_0\pi_1^2 - \frac{\gamma_1^2}{\gamma_0} \end{aligned} \tag{2.15}$$

where $\gamma_0, \gamma_1 > 0$. With the above equations, the parameters T and τ^* are generated by the following conditions.

Condition 2.2 *For* $\lambda \in (0, 1)$, *the pair* (τ^*, T) *with* $0 \leq \tau^* \leq T$ *satisfies the following conditions*

$$\begin{aligned} &\pi_1(0) = \frac{1}{\lambda}, \quad \pi_0(0) \in (\lambda, \frac{1}{\lambda}], \\ &\pi_0(T) = \lambda, \quad \pi_0(\theta) \leq \pi_1(\theta), \quad \forall \theta \in [0, \tau^*] \end{aligned} \tag{2.16}$$

where π_0, π_1 *denote solutions of systems (2.15).* □

Since any solution of the second equation in (2.15) with positive initial value will be decreasing and the righthand side has the negative constant term, we can always find a number $T > 0$ such that $\pi_0(T) = \lambda$ with $\pi_0(0) \in (\lambda, \frac{1}{\lambda}]$. Then we can also find a number $\tau^* \leq T$ such that $\pi_1(\theta) \geq \pi_0(\theta), \forall\theta \in [0, \tau^*]$.

We are ready to present one of the main results in this chapter.

Theorem 2.1 *Consider systems (2.12) and suppose Assumption 2.3 holds. Assume Condition 2.1, 2.2 hold. Suppose that the functions in transmission strategy (2.10) are designed as follows*

$$f_{\upsilon,0}(\upsilon, x) := -\rho_0(\upsilon) + \phi(x)$$
$$f_{\upsilon,1}(\upsilon, x, e) := -\rho_1(\upsilon) + \varphi(x) - \chi(e)$$

where ρ_0, ρ_1 are arbitrarily class-$\mathcal{K}$ functions, then the set $\mathscr{W} = \{\mathbf{0}\} \times \{\mathbf{0}\} \times \{\mathbf{0}\} \times \{0\} \times \{0, 1, 2\} \times \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ will be UGAS for systems (2.12). ☐

Proof of Theorem 2.1 Let $\tilde{\pi}_0 = \pi_0$ and $\tilde{\pi}_1 = \frac{\gamma_0}{\gamma_1}\pi_1$. Then Eq. (2.15) can be transformed into the following form

$$\begin{aligned}\dot{\tilde{\pi}}_0 &= -2L_0\tilde{\pi}_0 - \gamma_0\tilde{\pi}_0^2 - \gamma_0\\ \dot{\tilde{\pi}}_1 &= -2L_1\tilde{\pi}_1 - \gamma_1\tilde{\pi}_1^2 - \gamma_1.\end{aligned} \tag{2.17}$$

The conditions (2.16) can be transformed into the following forms

$$\begin{aligned}&\tilde{\pi}_1(0) = \frac{\gamma_0}{\gamma_1}\frac{1}{\lambda}, \quad \tilde{\pi}_0(0) \in (\lambda, \frac{1}{\lambda}]\\ &\tilde{\pi}_0(T) = \lambda, \qquad \gamma_1\tilde{\pi}_1(\theta) \geq \gamma_0\tilde{\pi}_0(\theta), \ \forall\theta \in [0, \tau^*].\end{aligned} \tag{2.18}$$

Choose the following function

$$U(z) := \begin{cases} V(x) + \gamma_l\tilde{\pi}_l(\tau)W_l^2(\kappa, e, s) + \upsilon, & z \in C_1 \cup C_2 \cup D_1 \cup D_2\\ V(x) + \lambda\gamma_0 W_l^2(\kappa, e, s) + \upsilon, & z \in C_3 \cup D_3\end{cases}$$

where $\tilde{\pi}_l$ are the solutions of (2.17) satisfying (2.18). The function U is locally Lipschitz continuous on $C \cup D \cup g(D)$ and there exist $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$ such that $\tilde{\alpha}_1(|z|_{\mathscr{W}}) \leq U(z) \leq \tilde{\alpha}_2(|z|_{\mathscr{W}})$ for all $z \in C \cup D$. For almost all $z \in C_1$, it holds that

$$\begin{aligned}\langle\nabla U(z), f\rangle \leq \frac{\partial V}{\partial x}f_1 + \gamma_1\dot{\tilde{\pi}}_1 W_1^2(\kappa, e, s) + 2L_1\gamma_1\tilde{\pi}_1 W_1(\kappa, e, s)\\ + 2\gamma_1\tilde{\pi}_1 W_1(\kappa, e, s)H_1(x) + \dot{\upsilon}.\end{aligned} \tag{2.19}$$

Using (C2) of Condition 2.1 and substituting $\dot{\tilde{\pi}}_1$ and $\dot{\upsilon}$ into (2.19), we can obtain

$$\begin{aligned}\langle \nabla U(z), f\rangle \leq & -\delta|(x,e,s)|^2 - \phi(x) - H_1^2(x) + \gamma_1^2 W_1^2(\kappa,e,s)\\ & - \gamma_1^2\tilde{\pi}_1^2 W_1^2(\kappa,e,s) - \gamma_1^2 W_1^2(\kappa,e,s)\\ & + 2\gamma_1\tilde{\pi}_1 W_1(\kappa,e,s)H_1(x) - \rho_0(\upsilon) + \phi(x)\\ \leq & -\delta|(x,e,s)|^2 - \rho_0(\upsilon).\end{aligned}$$

Similarly, for almost all $z \in C_2$, $\langle \nabla U(z), f\rangle \leq -\delta|(x,e,s)|^2 - \rho_0(\upsilon)$. For almost all $z \in C_3$, it holds by combining (C2) and (C3) of Condition 2.1 that

$$\langle \nabla U(z), f\rangle \leq -\delta|(x,e,s)|^2 - \rho_1(\upsilon).$$

Since ρ_0, ρ_1 are class-$\mathscr{K}$ functions, there exists a positive definite function α such that for almost all $z \in C$, we have

$$\dot{U} \leq -\alpha(U).$$

For $z \in D_1$, it holds that

$$\begin{aligned}U(g(z)) &= V(x) + \gamma_0\tilde{\pi}_0(\tau)W_0^2(\kappa,e,s)\\ &\leq V(x) + \gamma_1\tilde{\pi}_1(\tau)W_1^2(\kappa,e,s)\\ &= U(z).\end{aligned}$$

For $z \in D_2$, it is easy to derive

$$U(g(z)) \leq U(z).$$

For $z \in D_3$, it holds that

$$\begin{aligned}U(g(z)) &= V(x) + \gamma_1\tilde{\pi}_1(0)W_1^2(\kappa+1,e,h(\kappa,e))\\ &\leq V(x) + \frac{\gamma_0}{\lambda}W_1^2(\kappa+1,e,h(\kappa,e))\\ &= U(z)\end{aligned}$$

where we have used Assumption 2.3. As a result, $U(g(z)) \leq U(z)$ for all $z \in D$. Note that since of the existence of the waiting time T, there exist $c_1 > 0$ and $c_2 > 0$ such that for any $(t_1, j_1), (t_2, j_2) \in \mathrm{dom} z$, $j_2 - j_1 \leq c_1(t_2 - t_1) + c_2$. Therefore, systems (2.12) have persistent flow dynamics and UGAS of the set $\mathscr{W}$ is concluded by similar arguments to that of [9]. □

Remark 2.2 From the designed functions $f_{\upsilon,0}$, $f_{\upsilon,1}$ in Theorem 2.1, once a triggering condition $\upsilon \leq 0$ is satisfied, the sensors will sample the output and sent it to the controller. The variable υ will be subject to the following equation

$$\dot{\upsilon} = -\rho(\upsilon) + \phi(x).$$

Therefore, once υ arrives at the value 0, it may increase because of the nonnegative term $\phi(x)$. This mechanism may avoid the probability of periodic sampling.

2.4 Distributed Model-Based Dynamic Event-Triggered Control

In this section, we consider the distributed case. Consider the following plant consisting of N subsystems:

$$\dot{x}_i = \sum_{j=1} A_{ij} x_j + B_i u_i, \quad i \in Q := \{1, 2, \ldots, N\} \tag{2.20}$$

where $x_i \in \mathbb{R}^{n_{x_i}}$ is the state of i-th subsystem. A distributed model-based controller is given as follows to stabilize systems (2.20):

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{A}_{ii}\hat{x}_i + \hat{B}_i u_i, \\ u_i &= K_i \hat{x}_i \end{aligned} \tag{2.21}$$

where $\hat{A}_{ii}$ and $\hat{B}_i$ are the ideal parameters of system matrices A_{ii} and B_i. Each controller (2.21) uses a model that omits the information of other subsystems and thus is distributed. We assume that each subsystem sends its state x_i to the controller via a network $\mathcal{N}_i$ and there exist $\ell_i \in \mathbb{Z}_{\geq 0}$ nodes in the network $\mathcal{N}_i$. Let $\{t_k^i\}_{k=0}^{\infty}$ be the sampling sequence of network $\mathcal{N}_i$. The variable x_i can be divided into $x_i = (x_{i1}, x_{i2}, \ldots, x_{i\ell_i})$ correspondingly with $x_{ij} \in \mathbb{R}^{n_{x_{ij}}}$. In network $\mathcal{N}_i$, the transmission order of nodes is determined by the transmission protocols. The transmitted data x_{ij} at time t_k^i will arrive at the controller at $t_k^i + d_k^i$ where d_k^i is the transmission delay with $0 \leq d_k^i < t_{k+1}^i - t_k^i$. The sampling times of network $\mathcal{N}_i$ are determined by the following event-triggered transmission strategy

$$t_{k+1}^i := \inf\{t \geq t_k^i + T_i | \upsilon_i \leq 0\} \tag{2.22}$$

where $T_i > 0$ is the waiting time in network $\mathcal{N}_i$. The variable υ_i is subject to the following dynamic equation with initial value $\upsilon_i(t_0^i) > 0$:

$$\dot{\upsilon}_i = \begin{cases} f_{\upsilon_i,0}(\upsilon_i, x_i(t)), & t \in [t_k^i, t_k^i + T_i) \\ f_{\upsilon_i,1}(\upsilon_i, x_i(t), e_i(t)), & t \in [t_k^i + T_i, t_{k+1}^i) \end{cases}$$

where $e_i := \hat{x}_i - x_i \in \mathbb{R}^{n_{e_i}}$. The variable $\hat{x}_i$ updates at $t_k^i + d_k^i$ as

$$\hat{x}_i((t_k^i + d_k^i)^+) = e^{(\hat{A}_{ii}+\hat{B}_i K_i)d_k^i}\left((I_{n_{x_i}} - \Phi_i(k))\hat{x}_i(t_k^i) + \Phi_i(k)x_i(t_k^i)\right) \tag{2.23}$$

where $\Phi_i(k)$ is a block diagonal matrix. The equation $h_i(k, e_i) := (I_{n_{x_i}} - \Phi_i(k))e_i$ is called the transmission protocol equation of network $\mathcal{N}_i$. The reason that $\hat{x}_i$ updates based on (2.23) is same as the centralized case.

Some assumptions are also needed as presented below.

Assumption 2.4 The transmission delays satisfy $0 \le d_i \le \tau_i^* \le T_i$. ☐

Assumption 2.5 For each network $\mathcal{N}_i (i \in Q)$, the transmission protocols are UGES which means that there exist functions $W_i : \mathbb{Z}_{\ge 0} \times \mathbb{R}^{n_{e_i}} \to \mathbb{R}_{\ge 0}$ which is locally Lipschitz continuous in its second variable, constants $a_{i1}, a_{i2} > 0$, and constants $\lambda_i \in (0, 1)$ such that the following conditions hold

$$\begin{aligned} a_{i1}|e_i| \le W_i(k, e_i) \le a_{i2}|e_i| \\ W_i(k+1, h_i(k, e_i)) \le \lambda_i W_i(k, e_i). \end{aligned}$$

Moreover, there exist constants $M_i > 0$, $\lambda_{W_i} \ge 1$ such that for all $k \in \mathbb{Z}_{\ge 0}$ and almost all $e_i \in \mathbb{R}^{n_{e_i}}$, it holds that

$$\left|\frac{\partial W_i(k, e_i)}{\partial e_i}\right| \le M_i$$

and for all $k \in \mathbb{Z}_{\ge 0}$ and all $e_i \in \mathbb{R}^{n_{e_i}}$, we have

$$W_i(k+1, e_i) \le \lambda_{W_i} W_i(k, e_i).$$ ☐

The entire closed-loop systems can be modeled into the following hybrid system by using a similar modeling approach to the centralized case:

$$\begin{cases} \dot{z} \in F(z), & z \in C \\ z^+ \in G(z), & z \in D \end{cases} \tag{2.24}$$

where $z := (x, e, s, \upsilon, l, \tau, \kappa) \in \mathbb{R}^{n_z}$, $x := (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{n_x}$, $e := (e_1, e_2, \ldots, e_N) \in \mathbb{R}^{n_e}$, $s := (s_1, s_2, \ldots, s_N) \in \mathbb{R}^{n_s}$, $\upsilon := (\upsilon_1, \upsilon_2, \ldots, \upsilon_N) \in \mathbb{R}^N$, $l := (l_1, l_2, \ldots, l_N)$, $\tau := (\tau_1, \tau_2, \ldots, \tau_N)$, $\kappa := (\kappa_1, \kappa_2, \ldots, \kappa_N)$. The flow dynamics are subject to

$$z \in C \quad \begin{cases} \dot{x} = f_1(x, e) \\ \dot{e} = f_2(x, e) \\ \dot{s} = f_3(x, e, s) \\ \dot{\upsilon} = f_4(x, e, \upsilon) \\ \dot{l} = \mathbf{0} \\ \dot{\tau} = \mathbf{1}_N \\ \dot{\kappa} = \mathbf{0} \end{cases} \tag{2.25}$$

where the flow set is defined as

$$
\begin{aligned}
C &:= \bigcap_{i \in Q} \bigcup_{q \in \{1,2,3\}} C_{iq} \\
C_{i1} &:= \{z | \upsilon_i \in \mathbb{R}_{\geq 0}, l_i = 1, \tau_i \in [0, \tau_i^*], \kappa_i \in \mathbb{Z}_{\geq 0}\} \\
C_{i2} &:= \{z | \upsilon_i \in \mathbb{R}_{\geq 0}, l_i = 0, \tau_i \in [0, T_i], \kappa_i \in \mathbb{Z}_{\geq 0}\} \\
C_{i3} &:= \{z | \upsilon_i \in \mathbb{R}_{\geq 0}, l_i = 2, \tau_i \in [T_i, \infty), \kappa_i \in \mathbb{Z}_{\geq 0}\}
\end{aligned}
$$

and

$$
\begin{aligned}
f_1(x, e) &:= A^* x + B^* \sum_{i=1}^{N} e_i \\
f_2(x, e) &:= (f_{21}(x, e), \ldots, f_{2N}(x, e)) \\
f_3(x, e, s) &:= (f_{31}(x, e, s), \ldots, f_{3N}(x, e, s)) \\
f_4(x, e, v) &:= (f_{41}(x, e, v), \ldots, f_{4N}(x, e, v)) \\
f_{2i}(x, e) &:= (\hat{A}_{ii} - \Delta B_{i0}) e_i - \Delta A_{i0} x_i - E_i x \\
f_{3i}(x, e, s) &:= \begin{cases} A_{i0} s_i - \Delta A_{i0} x_i - B_i K_i e_i - E_i x, & z \in C_{i1} \\ 0, & z \in C_{i2} \cup C_{i3} \end{cases} \\
f_{4i}(x, e, v) &:= \begin{cases} f_{\upsilon_i, 0}(\upsilon_i, x_i), & z \in C_{i1} \cup C_{i2} \\ f_{\upsilon_i, 1}(\upsilon_i, x_i, e_i), & z \in C_{i3} \end{cases} \\
A^* &:= \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1N} \\ A_{21} & A_2 & \cdots & A_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ A_{N1} & \cdots & A_{NN-1} & A_N \end{pmatrix} \\
B^* &:= \operatorname{diag}\{B_1 K_1, \ldots, B_N K_N\} \\
E_i &:= \begin{pmatrix} A_{i1} & \cdots & A_{ii-1} & 0_{n_{x_i}} & A_{ii+1} & \cdots & A_{1N} \end{pmatrix} \\
A_i &:= A_{ii} + B_i K_i \\
A_{i0} &: = \hat{A}_{ii} + \hat{B}_i K_i \\
\Delta A_{i0} &:= A_{ii} + B_i K_i - \hat{A}_{ii} - \hat{B}_i K_i \\
\Delta B_{i0} &:= B_i K_i - \hat{B}_i K_i .
\end{aligned}
$$

Denote

$$
\begin{aligned}
D_{i1} &:= C_{i1} \\
D_{i2} &:= \{z | \upsilon_i = 0, l_i = 0, \tau_i = T_i, \kappa_i \in \mathbb{Z}_{\geq 0}\} \\
D_{i2} &:= \{z | \upsilon_i = 0, l_i = 2, \tau_i \in [T_i, \infty), \kappa_i \in \mathbb{Z}_{\geq 0}\}.
\end{aligned}
$$

Then the jump set is specified as

$$D := \bigcup_{i \in Q, q \in \{1,2,3\}} D_{iq}.$$

Let $\varGamma_i$ denote an N-dimensional diagonal matrix with the i-th diagonal element being zero and the other diagonal elements being 1 and let $\varGamma_i^*$ denote an n_e-dimensional block diagonal matrix consisting of N diagonal blocks as follows:

$$\mathrm{diag}\{I_{n_{e_1}}, \ldots, I_{n_{e_{i-1}}}, 0_{n_{e_i}}, I_{n_{e_{i+1}}}, \ldots, I_{n_{e_N}}\}.$$

The jump map is specified as

$$G(z) := \bigcup_{i \in Q} G_i(z)$$

where

$$G_i(z) := \begin{cases} \widetilde{G}_i(z), & z \in \bigcup\limits_{q \in \{1,2,3\}} D_{iq} \\ \emptyset, & \text{otherwise} \end{cases}$$

and

$$\widetilde{G}_i(z) := \begin{pmatrix} x \\ \varGamma_i^* e + (I_{n_e} - \varGamma_i^*) s \\ \varGamma_i^* s \\ v \\ \varGamma_i l \\ \tau \\ \kappa \end{pmatrix}, \quad z \in D_{i1},$$

$$\widetilde{G}_i(z) := \begin{pmatrix} x \\ e \\ s \\ v \\ l + (I_N - \varGamma_i) 1_N \\ \tau \\ \kappa \end{pmatrix}, \quad z \in D_{i2},$$

$$\widetilde{G}_i(z) := \begin{pmatrix} x \\ e \\ \Gamma_i^* s + (I_{n_e} - \Gamma_i^*)h(\kappa, e) \\ v \\ \Gamma_i l + (I_N - \Gamma_i)1_N \\ \Gamma_i \tau \\ \Gamma_i \kappa + (I_N - \Gamma_i)(\kappa + 1_N) \end{pmatrix}, \quad z \in D_{i3}.$$

Note that $h(\kappa, e) := (h_1(\kappa_1, e_1), \ldots, h_N(\kappa_N, e_N))$.

We assume that bounds on the model uncertainties are known, as characterized in the following assumption:

Assumption 2.6 The constants $\delta_{A_{i0}} > 0, \delta_{B_{i0}} > 0, i \in Q$ are such that the model uncertainties satisfy $|\Delta A_{i0}| \leq \delta_{A_{i0}}, |\Delta B_{i0}| \leq \delta_{B_{i0}}$. ☐

To analyze stability of the subsystems (e_i, s_i), consider the following functions

$$\begin{aligned} W_{i1}(\kappa_i, e_i, s_i) &: = \max\{W_i(\kappa_i, s_i), \frac{\lambda_i}{\lambda_{W_i}} W_i(\kappa_i, e_i)\}, \\ W_{i0}(\kappa_i, e_i, s_i) &: = W_i(\kappa_i, e_i) \\ W_{i2}(\kappa_i, e_i, s_i) &: = W_i(\kappa_i, e_i). \end{aligned}$$

Under Assumption 2.5, we can conclude the following lemma,

Lemma 2.2 *Suppose Assumption 2.5 holds. Then for all* $\kappa_i \in \mathbb{Z}_{\geq 0}, x \in \mathbb{R}^{n_x}$ *and almost all* $e_i \in \mathbb{R}^{n_{e_i}}, s_i \in \mathbb{R}^{n_{s_i}}$*, it holds that*

$$\left| \frac{\partial W_{il_i}(\kappa_i, e_i, s_i)}{\partial (e_i, s_i)} (f_{2i}(x, e), f_{3i}(x, e, s)) \right| \leq L_{il_i} W_{il_i} + H_{il_i}(x)$$

where $L_{i0} = L_{i2} := \frac{M_i(|\hat{A}_{ii}| + \delta_{B_{i0}})}{a_{i1}}$, $H_{il_i}(x) := M_i \delta_{A_{i0}} |x_i| + M_i |E_i x|$, *and*

$$L_{i1} := \max\left\{ \frac{M_i(|\hat{A}_{ii}| + \delta_{B_{i0}})}{a_{i1}}, \frac{M_i(\lambda_i |A_{i0}| + \lambda_{W_i} |B_i K_i|)}{\lambda_i a_{i1}} \right\}.$$ ☐

We will use the following conditions to design the event-triggered transmission strategy.

Condition 2.3 *Suppose that there exist constants* $\alpha_j > 0 (j = 1, 2)$*, positive definite functions* $\phi_i, \varphi_i : \mathbb{R}^{n_{x_i}} \to \mathbb{R}_{\geq 0}$ *and* $\chi_i : \mathbb{R}^{n_{e_i}} \to \mathbb{R}_{\geq 0} (i \in Q)$*, constants* $\delta > 0, \gamma_{i1} > 0, \gamma_{i0} = \gamma_{i2} > 0 (i \in Q)$*, and a continuously differentiable function* $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ *such that the following conditions hold:*

(C1) $\alpha_1 |x|^2 \leq V(x) \leq \alpha_2 |x|^2$ *for all* $x \in \mathbb{R}^{n_x}$,

(C2) for all $z \in C$*, it holds that*

$$\frac{\partial V}{\partial x} f_1(x,e) \le -\delta|(x,e,s)|^2 - \sum_{i=1}^{N} \phi_i(x_i) - \sum_{i=1}^{N} (H_{il_i}^2(x) - \gamma_{il_i}^2 W_{il_i}^2(\kappa_i, e_i, s_i)),$$

(C3) for all $x \in \mathbb{R}^{n_e}$, $e \in \mathbb{R}^{n_e}$ *and* $i \in Q$,

$$\begin{aligned} &H_{i0}^2(x) + \chi_i(e_i) + \phi_i(x_i) \\ &\ge \gamma_{i0}^2 W_i^2(\kappa_i, e_i) + 2\lambda_i \gamma_{i0} W_i(\kappa_i, e_i)(L_{i0} W_i(\kappa_i, e_i) + H_{i0}(x)) + \varphi_i(x_i). \end{aligned}$$ □

Remark 2.3 Essence of (C2) in Condition 2.3 is that x-subsystem is input-to-state stable with respect to e and different gains are assigned to each error state e_i, $i \in Q$.

The waiting time T_i and upper bound τ_i^* of transmission delays for each network $\mathcal{N}_i$ will also be determined by associating a pair of differential equations. Consider the following two differential equations for each $i \in Q$

$$\begin{aligned} \dot{\pi}_{i0} &= -2L_{i0}\pi_{i0} - \gamma_{i0}\pi_{i0}^2 - \gamma_{i0} \\ \dot{\pi}_{i1} &= -2L_{i1}\pi_{i1} - \gamma_{i0}\pi_{i1}^2 - \frac{\gamma_{i1}^2}{\gamma_{i0}} \end{aligned} \tag{2.26}$$

where $\gamma_{i0}, \gamma_{i1} > 0$.

The waiting time and upper bound of transmission delays will satisfy the following conditions.

Condition 2.4 *For* $\lambda_i \in (0, 1)$, *the pair* (τ_i^*, T_i) *with* $0 \le \tau_i^* \le T_i$ *satisfies the following conditions*

$$\begin{aligned} &\pi_{i1}(0) = \frac{1}{\lambda_i}, \quad \pi_{i0}(0) \in (\lambda_i, \frac{1}{\lambda_i}], \\ &\pi_{i0}(T_i) = \lambda_i, \quad \pi_{i0}(\theta) \le \pi_{i1}(\theta), \quad \forall \theta \in [0, \tau_i^*] \end{aligned}$$

where π_{i0}, π_{i1} *denote the solutions of (2.26).* □

We are ready to present another main theorem of this chapter.

Theorem 2.2 *Consider the systems (2.24) under Assumption 2.4, 2.5, 2.6. Suppose Condition 2.3, 2.4 hold, then the set* $\mathcal{W} := \{\mathbf{0}\} \times \{\mathbf{0}\} \times \{\mathbf{0}\} \times \{\mathbf{0}\} \times \{0, 1, 2\}^N \times \mathbb{R}_{\ge 0}^N \times \mathbb{Z}_{\ge 0}^N$ *is UGAS for systems (2.24) if the functions in transmission strategy (2.22) are designed as follows*

$$\begin{aligned} f_{\upsilon_i,0}(\upsilon_i, x_i) &:= -\rho_{i0}(\upsilon_i) + \phi_i(x_i) \\ f_{\upsilon_i,1}(\upsilon_i, x_i, e_i) &:= -\rho_{i1}(\upsilon_i) + \varphi_i(x_i) - \chi_i(e_i). \end{aligned}$$

Proof Consider the following function $U : C \cup D \to \mathbb{R}$:

$$U(z) := V(x) + \sum_{i=1}^{N} U_i$$

where

$$U_i(e_i, s_i, \upsilon_i, l_i, \tau_i, \kappa_i) := \begin{cases} \gamma_{il_i}\pi_{il_i}(\tau_i)W_{il_i}(\kappa_i, e_i, s_i) + \upsilon_i, & z \in C_{i1} \cup C_{i2} \cup D_{i1} \cup D_{i2} \\ \lambda_i \gamma_{i0} W_{il_i}(\kappa_i, e_i, s_i) + \upsilon_i, & z \in C_{i3} \cup D_{i3}. \end{cases}$$

Using Lemma 2.2, we can derive from Condition 2.3 that for all $z \in C$

$$\dot{U}(z) \leq -\delta|(x, e, s)|^2 - \sum_{i=1}^{N} \min\{\rho_{i0}(\upsilon_i), \rho_{i1}(\upsilon_i)\}$$

and using Assumption 2.5, we can conclude that for all $z \in D$ and $g \in G(z)$, $U(g) \leq U(z)$. Because each local network has a waiting time, systems (2.24) have persistent flow dynamics. The remaining proofs are similar to that of Theorem 2.1. □

Remark 2.4 The paper [13] considers decentralized dynamic event-triggered transmission strategy for nonlinear systems with output feedback. The difference is that the controller in [13] uses the zero-order-hold while this chapter considers model-based dynamic event-triggered control.

2.5 Example and Simulation

Consider the linearized model of the interconnected pendulum used in [26, 47, 49]. The matrix parameters are as follows

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2.9156 & 0 & -0.0005 & 0 \\ 0 & 0 & 0 & 1 \\ -1.6663 & 0 & 0.0002 & 0 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.0011 & 0 & 0.0005 & 0 \\ 0 & 0 & 0 & 0 \\ -0.0003 & 0 & -0.0002 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -0.0042 \\ 0 \\ 0.0167 \end{bmatrix}$$

$$B_2 = B_1, \quad A_{22} = A_{11}, \quad A_{21} = A_{12}.$$

Such system is open-loop unstable. To stabilize this system, the controller gains are chosen as follows

$$K_1 = \begin{bmatrix} 11396 \ 7196.2 \ 573.96 \ 1199.0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 29241 \ 18135 \ 2875.3 \ 3693.9 \end{bmatrix}.$$

If each network has two nodes and the transmission protocols are Try-Once-Discard protocols then $a_{11} = a_{12} = a_{21} = a_{22} = 1$, $M_1 = M_2 = 1$ and $\lambda_1 = \lambda_2 = \sqrt{\frac{1}{2}}$, $\lambda_{W_1} = \lambda_{W_2} = 1$ (refer to [29, 64]). We can compute $L_{10} = 3.3582$, $L_{11} = 561.0889$, $L_{20} = 3.3582$, $L_{21} = 1441.7$, $H_{il_i}(x) = 0.0013|x|$ in the absence of disturbances. Choose Lyapunov function $V(x) := x^T P x$, parameter $\delta = 0.001$, $\phi_i(x_i) = 0.02|x_i|^2$, and transform (C2) of Condition 2.3 into linear matrix inequality, then we can obtain $\gamma_{10} = \gamma_{20} = 8.3$ by LMI toolbox in MATLAB (these two gains can be optimized by function mincx in LMI toolbox). We can also obtain the gains $\gamma_{11} = \gamma_{21} = 12$. Therefore, we can choose $\rho_{i0}(\upsilon_i) = \rho_{i1}(\upsilon_i) = -2\upsilon_i$, $\varphi_i(x_i) = 0.02|x_i|^2$ and $\chi_i(e_i) = 160|e_i|^2$ such that (C3) of Condition 2.3 holds. We run simulations with initial state $x_1(0) = x_2(0) = (0.02, 0.02, -0.02, -0.02)$ under $T_i = 0.005$ and constant delays $\tau_i^* = 0.001$ for two cases: one uses model-based controllers for both subsystems and the other uses controller based on zero-order-hold for both subsystems. Both cases use Try-Once-Discard protocols in network $\mathscr{N}_1$ and $\mathscr{N}_2$. Figures 2.1, 2.2, 2.3 illustrate the state trajectories and release intervals between two sampling points in both networks under model-based controller and Figs. 2.4, 2.5, 2.6 illustrate the state trajectories and release intervals in both networks with controller based on zero-order-hold. The simulations show that data transmission is sparse under model-based event-triggered transmission strategy while data transmission is dense under controller with zero-order-hold, and thus the proposed transmission strategy greatly reduces data transmission in network compared with the controller based on zero-order-hold. This illustrates the effectiveness of the proposed controllers in this chapter.

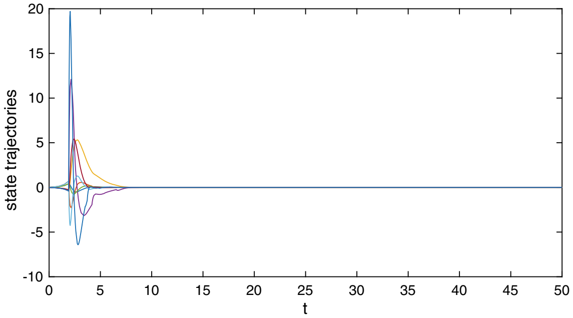


Fig. 2.1 State trajectories based on model-based controller

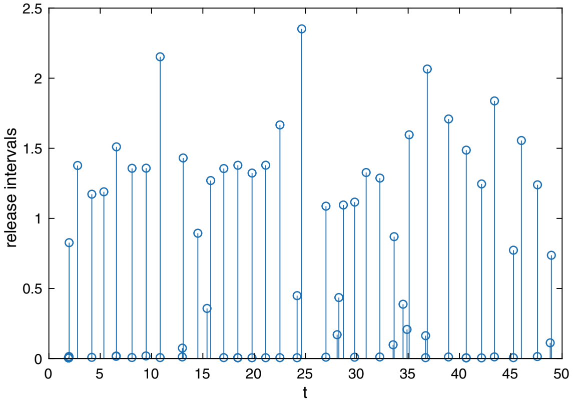


Fig. 2.2 Release intervals in network $\mathscr{N}_1$ under model-based controller

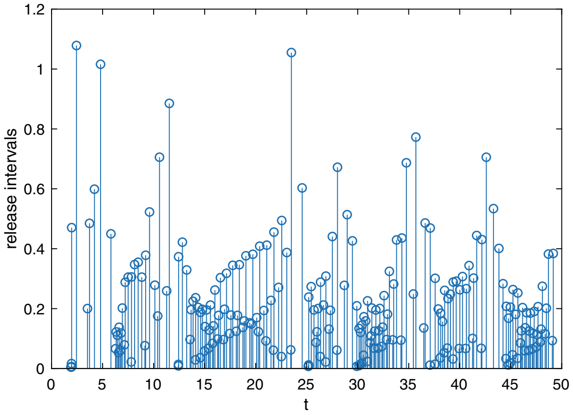


Fig. 2.3 Release intervals in network $\mathscr{N}_2$ under model-based controller

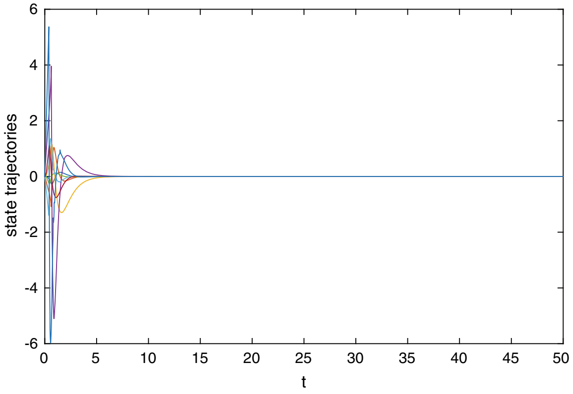


Fig. 2.4 State trajectories with controller based on zero-order-hold

Fig. 2.5 Release interval of network $\mathcal{N}_1$ with controller based on zero-order-hold

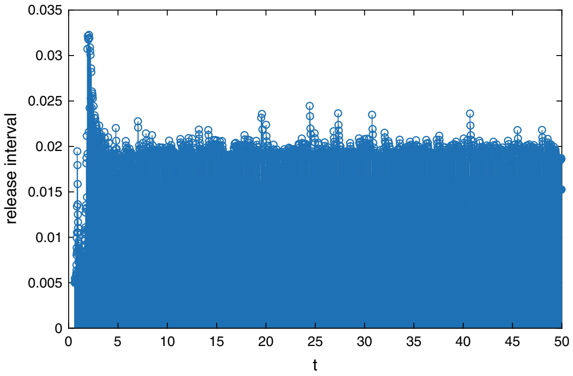
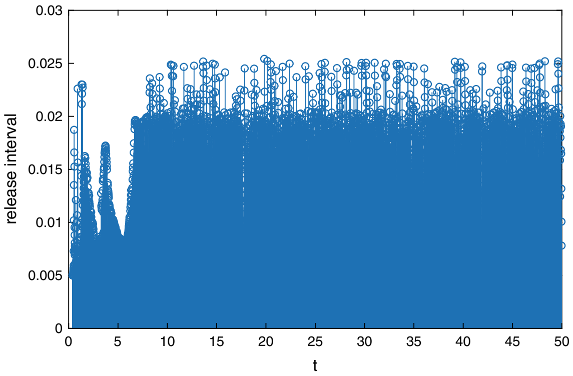


Fig. 2.6 Release interval of network $\mathcal{N}_2$ with controller based on zero-order-hold



2.6 Conclusions

The results established in the previous sections consider only linear systems and state feedback. It is not hard to extend the proposed results to nonlinear systems with special forms, for example,

$$\dot{x} = Ax + \tilde{f}(x) + Bu. \tag{2.27}$$

For systems (2.27), a model-based controller can be given as follows

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, \quad u = K\hat{x}. \tag{2.28}$$

Following the same approaches as previous sections, we can also design the centralized and distributed dynamic event-triggered transmission strategies for (2.27) and (2.28). Besides, suppose that systems (2.27) can be decomposed into subsystems. If some subsystems can be stabilized by state feedback and the other can be stabilized by output feedback, then for state feedback subsystems, we can design model-based dynamic event-triggered transmission strategy and for output feedback subsystems, we can design dynamic event-triggered transmission strategy based on zero-order-hold as done in [13].

Chapter 3
Periodic Event-Triggered Control for Decentralized Linear Systems with Quantization Effects and External Disturbances

3.1 Introduction

Quantization effects are almost inevitable in a sampled-data control system and the possible saturation of the quantizer may lead to instability of the closed-loop networked control system [5, 43, 45, 77, 85, 92]. The co-design of the event-triggered transmission strategy and quantization strategy has become a challenging and hot topic in recent years [1, 17, 18, 25, 39, 41, 50, 52, 53, 74, 81, 94, 95, 109, 110].

In [18], the authors consider model-based event-triggered controllers for linear systems with quantization effects and delays in the communication network. Asymptotic stability is obtained with the logarithmic quantizer in [18] while the quantizer is assumed to have an infinite quantization region. In [52], the authors investigate event-triggered transmission strategies for nonlinear systems with finite quantization region in the absence of external disturbances, and the state can be steered to the origin by using dynamic quantizers. As for the case with external disturbances, the quantizer may be saturated and the system may become unstable [45]. In [17], a decentralized periodic event-triggered transmission strategy is considered for linear systems with external disturbances in the presence of signal quantizations while the quantizer saturation is not addressed. The event-triggered transmission strategies in [53] use the estimation of external disturbances and input-to-state stability is obtained. The authors in [109] consider event-triggered sliding-mode controllers, and use zoom in and zoom out strategies for the dynamic quantizers to avoid saturation of the quantizers. In [1], the authors propose a hybrid algorithm for the dynamic quantizer when considering a dynamic event-triggered transmission strategy for nonlinear systems with external disturbances. The system is input-to-state stable and the quantizer saturation is avoided.

In this chapter, a hybrid algorithm of the dynamic quantizer is given for decentralized periodic event-triggered control systems with quantization effects in the presence of external disturbances. The quantizer is assumed to have a finite quantization region. The entire closed-loop system is modeled into a hybrid dynamical system [22]. Without considering disturbances, asymptotic stability can be obtained for the

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hybrid system and sufficient stability conditions are also derived. In the presence of external disturbances, $\mathscr{L}_2$ gain performance is guaranteed.

3.2 Preliminaries

For a locally Lipschitz continuous function $U : \mathbb{R}^n \to \mathbb{R}$ and a vector $\nu \in \mathbb{R}^n$, the Clarke's generalized derivative of U along the direction of vector ν is defined as:

$$U^\circ(x, \nu) = \limsup_{y \to x, h \to 0^+} \frac{U(y + h\nu) - U(y)}{h}.$$

The following lemma can be found in [46].

Lemma 3.1 *Let R_1 and R_2 be two locally Lipschitz continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$. For the function $R : \mathbb{R}^n \to \mathbb{R}$ defined by $R(y) := \max\{R_1(y), R_2(y)\}$, define sets $S_1 := \{y | R_1(y) > R_2(y)\}$, $S_2 := \{y | R_1(y) < R_2(y)\}$ and $S_3 := \{y | R_1(y) = R_2(y)\}$. Then it holds that (I) for $y \in S_1$, $R^\circ(y, \nu) = R_1^\circ(y, \nu)$; (II) for $y \in S_2$, $R^\circ(y, \nu) = R_2^\circ(y, \nu)$; (III) for $y \in S_3$, $R^\circ(y, \nu) \le \max\{R_1^\circ(y, \nu), R_2^\circ(y, \nu)\}$.*

3.3 Model Description

Consider the following decentralized system

$$\dot{x}_i = \sum_{j=1}^{N} A_{ij} x_j + B_i u_i + E_i w_i, \quad i \in Q := \{1, 2, \ldots, N\} \tag{3.1}$$

where $x_i \in \mathbb{R}^{n_{x_i}}$ is the state of the i-th subsystem, $u_i \in \mathbb{R}^{n_{u_i}}$ is the control input of the i-th subsystem, and $w_i \in \mathbb{R}^{n_{w_i}}$ is the external disturbance. To stabilize the system (3.1), use the following sampled-data controller for each subsystem

$$u_i = K_i \hat{x}_i, \quad t \in [kT_i, (k+1)T_i) \tag{3.2}$$

where $T_i > 0$ is a constant, $kT_i (k \in \mathbb{Z}_{\geq 0})$ are the detection instants, and $\hat{x}_i \in \mathbb{R}^{n_{x_i}}$ is a variable at the controller side.

For each subsystem, at the detecting instant kT_i, a triggering condition will be checked to determine whether the state is sampled through a quantizer and then sent to the controller. The following dynamic quantizer will be used:

$$q_{\mu_i}(x_i) := \mu_i q_i \left(\frac{x_i}{\mu_i} \right)$$

where q_i is a static quantizer function. The static quantizer functions satisfy the following assumption.

Assumption 3.1 There exist constants M_i and Δ_i with $M_i > \Delta_i > 0$ such that the following implication holds for each $i \in Q$:

$$|x_i| \le M_i \Rightarrow |q_i(x_i) - x_i| \le \Delta_i.$$

□

M_i characterizes the quantization region of the quantizer q_i and Δ_i is the quantization accuracy. To ensure that the static quantizer is not saturated, we need to adjust the quantizer coefficient μ_i dynamically.

Denote $e_i := \hat{x}_i - x_i$. For the i-th subsystem, at each detecting instant, once the following triggering condition is satisfied

$$|e_i| \ge \sigma_i |x_i|, \tag{3.3}$$

the quantizer coefficient μ_i will be updated as $g_{\mu_i,i}(x_i, \eta_i)$, which is defined as [92]:

$$g_{\mu_i,i}(x_i, \eta_i) := \begin{cases} \max\{\frac{|x_i|}{L_i}, \eta_i\}, & \frac{|x_i|}{\mu_i} \le L_{in}^i \\ \frac{|x_i|}{L_i}, & \frac{|x_i|}{\mu_i} \ge L_{out}^i \\ \mu_i, & L_{in}^i < \frac{|x_i|}{\mu_i} < L_{out}^i \end{cases} \tag{3.4}$$

where L_{in}^i, L_{out}^i and L_i are constants satisfying $0 < L_{in}^i < L_i < L_{out}^i \le M_i$ and each variable η_i satisfies the following ordinary differential equation

$$\dot{\eta}_i = -c_i \eta_i$$

with positive initial value. After the quantizer coefficient is updated, the state x_i is quantized according to the updated coefficient, and then the quantized state $q_{\mu_i}(x_i)$ is sent to the controller via the communication network. Once the controller receives the quantized state, the variable $\hat{x}_i$ will be updated as $q_{\mu_i}(x_i)$. Between two consecutive detecting instants, $\hat{x}_i$ is subject to the following equation

$$\dot{\hat{x}}_i = \tilde{f}_i(\hat{x}_i, u_i).$$

We consider two cases in this chapter. One case is that the controller in each subsystem uses a zero-order-hold. In this case, we have $\tilde{f}_i(\hat{x}_i, u_i) := 0$ for each $i \in Q$. The other case is that the controller in each subsystem is a model-based controller. In this case, $\tilde{f}_i(\hat{x}_i, u_i) := A_{ii}\hat{x}_i + B_i u_i$ for each $i \in Q$.

Introduce a logic variable $p_i \in \{0, 1\}$ and a timer variable $\tau_i \in [0, T_i]$ for each local network. The timer variable is used to generate the periodic detecting signal and the logic variable l_i is used to identify whether a sampling happens. Let $x := (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{n_x}$, $e := (e_1, e_2, \ldots, e_N) \in \mathbb{R}^{n_e}$, $\eta := (\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^N$, $\mu := (\mu_1, \mu_2, \ldots, \mu_N) \in \mathbb{R}^N$, $\tau := (\tau_1, \tau_2, \ldots, \tau_N) \in \mathbb{R}^N$, $p := (p_1, p_2, \ldots, p_N) \in \{0, 1\}^N$, and $w := (w_1, w_2, \ldots, w_N) \in \mathbb{R}^{n_w}$. Denote $z := (x, e, \eta, \mu, \tau, p)$. Then the entire closed-loop system can be formulated into the following hybrid system:

$$\begin{cases} \dot{z} \in F(z, w), & z \in C \\ z^+ \in G(z), & z \in D. \end{cases} \tag{3.5}$$

The flow set C is defined as $C := \bigcap_{i=1}^N C_i$ where

$$C_i := \{z|\tau_i \in [0, T_i],\ p_i = 0,\ \mu_i \in \mathbb{R}_{\geq 0},\ \eta_i \in \mathbb{R}_{\geq 0}\}.$$

The jump set D is defined as $D := \bigcup_{i=1}^N D_i$ where

$$\begin{aligned} D_i &:= \bigcup_{q=1}^4 D_{iq} \\ D_{i1} &:= \{z|\tau_i = T_i,\ p_i = 0,\ \mu_i \in \mathbb{R}_{\geq 0},\ \eta_i \in \mathbb{R}_{\geq 0},\ |e_i| > \sigma_i|x_i|\} \\ D_{i2} &:= \{z|\tau_i = T_i,\ p_i = 1,\ 0 \leq \mu_i \leq \max\{\eta_i, \frac{|x_i|}{L_{in}^i}\},\ \eta_i \in \mathbb{R}_{\geq 0},\ |e_i| \geq \sigma_i|x_i|\} \\ D_{i3} &:= \{z|\tau_i = T_i,\ p_i = 0,\ \mu_i \in \mathbb{R}_{\geq 0},\ \eta_i \in \mathbb{R}_{\geq 0},\ |e_i| < \sigma_i|x_i|\} \\ D_{i4} &:= \{z|\tau_i = T_i,\ p_i = 0,\ \mu_i \in \mathbb{R}_{\geq 0},\ \eta_i \in \mathbb{R}_{\geq 0},\ |e_i| = \sigma_i|x_i|\}. \end{aligned}$$

The flow dynamics are subject to the following equation:

$$z \in C \quad \begin{cases} \dot{x} = M_{11}x + M_{12}e + M_{13}w \\ \dot{e} = M_{21}x + M_{22}e + M_{23}w \\ \dot{\eta} = -c\eta \\ \dot{\mu} = 0 \\ \dot{\tau} = 1_N \\ \dot{p} = 0 \end{cases}$$

where c is a diagonal matrix with the i-th diagonal element being c_i and

$$M_{11} := \begin{pmatrix} A_{11} + B_1K_1 & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} + B_2K_2 & \cdots & A_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} + B_NK_N \end{pmatrix}$$

$$M_{12} := \begin{pmatrix} B_1K_1 \\ B_2K_2 \\ \vdots \\ B_NK_N \end{pmatrix}, \quad M_{13} := \begin{pmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & E_N \end{pmatrix}.$$

If a zero-order-hold is used by each local controller, then we have $M_{21} := -M_{11}$, $M_{22} := -M_{12}$, $M_{23} := -M_{13}$. If a model-based controller is used for each subsystem, then we have

$$M_{21} : = \begin{pmatrix} 0 & A_{12} & \cdots & A_{1N} \\ A_{21} & 0 & \cdots & A_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ A_{N1} & A_{N2} & \cdots & 0 \end{pmatrix}, \quad M_{22} := \begin{pmatrix} A_{11} \\ A_{22} \\ \vdots \\ A_{NN} \end{pmatrix}$$

$$M_{23} : = -M_{13}.$$

The jump map is given as $G(z) := \bigcup_{i=1}^{N} g_i(z)$ where g_i is defined as:

$$g_i(z) := \begin{cases} \begin{pmatrix} x \\ e \\ \eta \\ \Gamma_i \mu + (I_N - \Gamma_i) g_\mu(x, \eta) \\ \tau \\ \Gamma_i p + (I_N - \Gamma_i) 1_N \end{pmatrix}, & z \in D_{i1} \\ \begin{pmatrix} x \\ \Gamma_i^* e + (I_{n_e} - \Gamma_i^*) v \\ \eta \\ \mu \\ \Gamma_i \tau \\ \Gamma_i p \end{pmatrix}, & z \in D_{i2} \\ \begin{pmatrix} x \\ e \\ \eta \\ \mu \\ \Gamma_i \tau \\ p \end{pmatrix}, & z \in D_{i3} \\ \begin{pmatrix} x \\ e \\ \eta \\ \Gamma_i \mu + (I_N - \Gamma_i) g_\mu(x, \eta) \\ \tau \\ \Gamma_i p + (I_N - \Gamma_i) 1_N \end{pmatrix} \bigcup \begin{pmatrix} x \\ e \\ \eta \\ \mu \\ \Gamma_i \tau \\ p \end{pmatrix}, & z \in D_{i4} \\ \emptyset, \ \text{otherwise} \end{cases}$$

with

$$v := \{(v_1, v_2, \ldots, v_N) : |v_i| \leq \Delta_i \mu_i,\ i \in Q\}$$
$$g_\mu(x, \eta) := (\bar{g}_{\mu_1,1}(x_1), \bar{g}_{\mu_2,2}(x_2), \ldots, \bar{g}_{\mu_N,N}(x_N))$$

and

$$\bar{g}_{\mu_i,i}(x_i, \eta_i) := \begin{cases} \max\{\frac{|x_i|}{L_i}, \eta_i\}, & |x_i| < L^i_{in}\mu_i \\ \frac{|x_i|}{L_i}, & |x_i| > L^i_{out}\mu_i \\ \mu_i, & L^i_{in}\mu_i < |x_i| < L^i_{out}\mu_i \\ \{\max\{\frac{|x_i|}{L_i}, \eta_i\}, \mu_i\}, & |x_i| = L^i_{in}\mu_i,\ \mu_i > 0 \\ \{\frac{|x_i|}{L_i}, \mu_i\}, & |x_i| = L^i_{out}\mu_i,\ \mu_i > 0 \\ \{0, \eta_i\}, & x_i = 0,\ \mu_i = 0. \end{cases}$$

For the case $w = 0$, the data of system (3.5) satisfies the hybrid basic conditions [22]. In addition, we note that in system (3.5), $\mu_i = 0$ is allowed while in practical systems, according to the definition of functions $g_{\mu_i,i}$ in (3.4), μ_i will always be positive if the initial value of μ_i is set to be positive. Therefore, the hybrid model (3.5) can generate more solutions than the practical sampled-data systems. We can deduce stability of the practical systems from stability of the hybrid system (3.5).

3.4 Stability and Performance

In this section, we will analyze stability and performance of system (3.5) by constructing a Lyapunov function.

Define

$$F_i := \begin{pmatrix} a_{1i} & a_{2i} & \cdots & a_{Ni} \end{pmatrix}$$

where a_{ii} is a n_{x_i}-dimensional identity matrix and $a_{ji}(j \neq i)$ is a $n_{x_i} \times n_{x_j}$-dimensional zero matrix. Denote $H_i := F_i M_{21}$.

Next, we will present a condition for the x subsystem.

Condition 3.1 *There exist a continuously differentiable function* $V(x) := x^T P x$ *with* $P > 0$*, constants* $\varepsilon > 0$*,* $\gamma_i > 0 (i \in Q)$ *and* $\delta_i > 0$ *such that the following inequalities hold*

$$\langle \nabla V(x), M_{11}x + M_{12}e\rangle \le -\varepsilon(|x|^2 + |e|^2) + \sum_{i=1}^{N}(\gamma_i^2|e_i|^2 - x^T H_i^T H_i x - \delta_i x^T x). \tag{3.6}$$

Moreover, there exist constants $\rho_i > 0, b_i > 0$ *and* $\lambda_i > 1$ *such that the following matrix inequality holds*

$$\begin{pmatrix} -H_i^T H_i - \delta_i + \rho_i S_i & \rho_i F_i^T B_i K_i \\ * & \gamma_i^2 - b_i \gamma_i \frac{1}{\lambda_i} \end{pmatrix} \le 0 \tag{3.7}$$

where $*$ *denotes an appropriate matrix in the symmetric matrix and* $S_i := F_i^T F_i M_{11} + M_{11}^T F_i^T F_i + b_i F_i^T F_i$. ☐

Consider the following ordinary differential equation for each $i \in Q$

$$\dot{\phi}_i = -2L_i\phi_i - \gamma_i\phi_i^2 - \gamma_i \tag{3.8}$$

where $L_i > 0$. If $\tilde{f}(x_i, u_i) = 0$ for each $i \in Q$, then $L_i := |B_i K_i|$. If $\tilde{f}(x_i, u_i) = A_{ii}\hat{x}_i + B_i u_i$ for each $i \in Q$, then $L_i := |A_{ii}|$.

Let ϕ_i be a solution of (3.8) with initial value $\phi_i(0) := \lambda_i \in (1, \infty)$. T_i^* is a positive number such that $\phi_i(T_i^*) := \frac{1}{\lambda_i}$.

Next, we will present one of the main theorems.

Theorem 3.1 *Consider system (3.5) with* $w = 0$ *and assume Condition 3.1 holds. Suppose* $T_i \le T_i^*$ *and the following inequalities hold for each* $i \in Q$*:*

$$0 \le \sigma_i \le \sqrt{\frac{\rho_i}{\gamma_i\lambda_i}}, \quad L_{in}^i \ge \Delta_i\sqrt{\frac{\gamma_i\lambda_i}{\rho_i}}, \quad c_i > \lambda_i\gamma_i.$$

Then the set $\mathscr{W} := \{0\} \times \{0\} \times \mathbb{R}_{\ge 0}^N \times [0, T_i]^N \times \{0, 1\}^N$ *is uniformly globally asymptotically stable for system (3.5).* ☐

Remark 3.1 The analytical expression of the parameters T_i^* can be found in [9].

Proof of Theorem 3.1 Construct the following function for system (3.5)

$$U(z) := x^T P x + \sum_{i=1}^{N} V_i(z)$$

where the variable x is part of the variable z, and

$$V_i(z) := \max\{\rho_i|x_i|^2, d_i\eta_i^2, \gamma_i\phi_i(\tau_i)|e_i|^2\}.$$

and $d_i > 0$.

For each $z \in C$ and $f \in F(z, 0)$, we can derive

$$U^{\circ}(z, f) \leq \sum_{i=1}^{N} \left(-\varepsilon|x_i|^2 - \varepsilon|e_i|^2 + \gamma_i^2|e_i|^2 - x^T H_i^T H_i x - \delta_i x^T x + \dot{V}_i\right). \quad (3.9)$$

We discuss all the cases for each $i \in Q$.

Case I: $\rho_i|x_i|^2 > \max\{d_i\eta_i^2, \gamma_i\phi_i(\tau_i)|e_i|^2\}$

For this case, using Lemma 3.1, we can derive

$$\begin{aligned} \dot{V}_i &= 2\rho_i x_i^T(F_i M_{11} x + B_i K_i e_i) \\ &= \rho_i x^T(F_i^T F_i M_{11} + M_{11}^T F_i^T F_i)x + 2\rho_i x^T F_i^T B_i K_i e_i. \end{aligned} \quad (3.10)$$

Using (3.7) and (3.10) yields that

$$\begin{aligned} &-\varepsilon|x_i|^2 - \varepsilon|e_i|^2 + \gamma_i^2|e_i|^2 - x^T H_i^T H_i x - \delta_i x^T x + \dot{V}_i \\ &\leq -\varepsilon|x_i|^2 - \varepsilon|e_i|^2 + \gamma_i^2|e_i|^2 - x^T H_i^T H_i x - \delta_i x^T x + \dot{V}_i + b_i\rho_i|x_i|^2 - b_i\gamma_i\frac{1}{\lambda_i}|e_i|^2 \\ &\leq -\varepsilon|x_i|^2 - \varepsilon|e_i|^2 \\ &\leq -\alpha_i^1(|x_i|^2 + |e_i|^2 + \eta_i^2) \end{aligned}$$

for some $\alpha_i^1 > 0$.

Case II: $d_i\eta_i^2 > \max\{\rho_i|x_i|^2, \gamma_i\phi_i(\tau_i)|e_i|^2\}$

For this case, we can derive

$$\begin{aligned} \gamma_i^2|e_i|^2 - x^T H_i^T H_i x - \delta_i x^T x + \dot{V}_i &\leq \gamma_i^2|e_i|^2 - x^T H_i^T H_i x - \delta_i x^T x - d_i c_i \eta_i^2 \\ &\leq (\gamma_i\lambda_i - c_i)d_i\eta_i^2 \\ &\leq -\alpha_i^2(|x_i|^2 + |e_i|^2 + \eta_i^2) \end{aligned}$$

for some $\alpha_i^2 > 0$.

Case III: $\gamma_i\phi_i(\tau_i)|e_i|^2 > \max\{\rho_i|x_i|^2, d_i\eta_i^2\}$

For this case, we can derive

$$\begin{aligned} &-\varepsilon|x_i|^2 - \varepsilon|e_i|^2 + \gamma_i^2|e_i|^2 - x^T H_i^T H_i x - \delta_i x^T x + \dot{V}_i \\ &\leq -\varepsilon|x_i|^2 - \varepsilon|e_i|^2 + \gamma_i^2|e_i|^2 - x^T H_i^T H_i x - \delta_i x^T x \\ &\quad + \gamma_i\dot{\phi}_i(\tau_i)|e_i|^2 + \gamma_i\phi_i(\tau_i)2|e_i|(L_i|e_i| + |H_i x|) \\ &\leq -\varepsilon|x_i|^2 - \varepsilon|e_i|^2 - \delta_i x^T x \\ &\leq -\alpha_i^3(|x_i|^2 + |e_i|^2 + \eta_i^2) \end{aligned}$$

for some $\alpha_i^3 > 0$.

Similarly, for the other cases, by using Lemma 3.1, we can also derive

$$-\varepsilon|x_i|^2 - \varepsilon|e_i|^2 + \gamma_i^2|e_i|^2 - x^T H_i^T H_i x - \delta_i x^T x + \dot{V}_i \leq -\alpha_i^4(|x_i|^2 + |e_i|^2 + \eta_i^2)$$

for some $\alpha_i^4 > 0$.

As a result, we have shown that for any $z \in C$ and all $f \in F(z)$, it holds that

$$U^\circ(z, f) \leq -\alpha U \tag{3.11}$$

for some $\alpha > 0$.

For $z \in D_i$, we discuss four different cases.

(I) $z \in D_{i1}$

For this case, x, e, η, τ keep unchanged and thus $U(g) = U(z)$ for all $g \in G(z)$.

(II) $z \in D_{i2}$

For this case, it holds for $g \in G(z)$ that

$$\begin{aligned} V_i(g) &= \max\{\rho_i|x_i|^2, d_i\eta_i^2, \gamma_i\lambda_i\Delta_i^2\mu_i^2\} \\ &\leq \max\{\rho_i|x_i|^2, d_i\eta_i^2, \gamma_i\phi_i(T_i)|e_i|^2\} \\ &= V_i(z) \end{aligned}$$

where we have used the following inequality

$$\begin{aligned} \gamma_i\lambda_i\Delta_i^2\mu_i^2 &\leq \gamma_i\lambda_i\Delta_i^2\left(\max\{\eta_i, \frac{|x_i|}{L_{in}}\}\right)^2 \\ &\leq \max\{\rho_i|x_i|^2, \gamma_i\lambda_i\Delta_i^2\eta_i^2\} \end{aligned}$$

and d_i is chosen to be larger than $\gamma_i\lambda_i\Delta_i^2$. As a result, we have $U(g) \leq U(z)$ for all $g \in G(z)$.

(III) $z \in D_{i3}$

For this case, it holds that

$$\begin{aligned} V_i(g) &= \max\{\rho_i|x_i|^2, d_i\eta_i^2, \gamma_i\lambda_i|e_i|^2\} \\ &\leq \max\{\rho_i|x_i|^2, d_i\eta_i^2\} \\ &\leq V_i(z) \end{aligned}$$

where $g \in G(z)$. As a result, we have $U(g) \leq U(z)$ for all $g \in G(z)$.

Similarly, for $z \in D_{i4}$, we can also derive $U(g) \leq U(z)$ for all $g \in G(z)$. Therefore, for all $z \in D$, we have

$$U(g) \leq U(z) \tag{3.12}$$

for all $z \in G(z)$. Combining (3.11), (3.12) and noting that the system (3.1) satisfies the average dwell time constraint, we conclude $\mathscr{W}$ is uniformly globally asymptotically stable. □

Next, we will analyze $\mathscr{L}_2$ gain performance of system (3.5) in the presence of disturbances.

Condition 3.2 *There exist a continuously differentiable function* $V(x) := x^T Px$ *with* $P > 0$*, constants* $\varepsilon > 0, \theta > 0, \vartheta > 0$*, and* $\gamma_i > 0 (i \in Q)$ *such that the following inequality holds:*

$$\langle \nabla V(x), M_{11}x + M_{12}e \rangle \leq -\varepsilon(|x|^2 + |e|^2) + \sum_{i=1}^{N} (\gamma_i^2 |e_i|^2 - \widetilde{H}_i^2(x, w_i) - \delta_i(|x|^2 + |w_i|^2)) \\ - \theta(|x|^2 - \vartheta^2 |w|^2) \tag{3.13}$$

where $\widetilde{H}_i(x, w_i) := |F_i M_{21}x - E_i w_i|$.

Moreover, there exist constants $\rho_i > 0, b_i > 0$ *and* $\lambda_i > 1$ *such that the following matrix inequality holds*

$$\begin{pmatrix} -H_i^T H_i - \delta_i + \rho_i S_i & \Phi_i & \rho_i F_i^T B_i K_i \\ * & -\delta_i - E_i^T E_i & 0 \\ * & * & \gamma_i^2 - b_i \gamma_i \frac{1}{\lambda_i} \end{pmatrix} \leq 0 \tag{3.14}$$

where $*$ *denotes an appropriate matrix in the symmetric matrix and*

$$S_i := F_i^T F_i M_{11} + M_{11}^T F_i^T F_i + b_i F_i^T F_i$$
$$\Phi_i := M_{21}^T F_i^T E_i + \rho_i F_i^T E_i.$$

□

Theorem 3.2 *Consider system (3.5) and assume Condition 3.2 holds. Suppose* $T_i \leq T_i^*$ *and the following inequalities hold for each* $i \in Q$*:*

$$0 \leq \sigma_i \leq \sqrt{\frac{\rho_i}{\gamma_i \lambda_i}}, \quad L_{min}^i \geq \Delta_i \sqrt{\frac{\gamma_i \lambda_i}{\rho_i}}, \quad c_i > \lambda_i \gamma_i.$$

Then system (3.5) is $\mathscr{L}_2$ *stable with* $\mathscr{L}_2$ *gain* ϑ *to the disturbance* w *in the sense that the following inequality holds for all* $t \geq 0$

$$\left(\int_0^t |x(s,k)|^2 ds \right)^{\frac{1}{2}} \leq \beta(|(x(0), e(0), \eta(0))|) + \vartheta \left(\int_0^t |w(s,k)|^2 ds \right)^{\frac{1}{2}} \tag{3.15}$$

where β *is a class-*$\mathscr{K}_\infty$ *function.* □

Proof of Theorem 3.2 Using Lyapunov function same as that in the proof of Theorem 3.1, we can derive that for any $z \in C$ and all $f \in F(z, w)$, the following inequality holds

$$U^\circ(z, f) \leq -\alpha U - \theta(|x|^2 - \vartheta^2|w|^2),$$

and for all $z \in D$ and $g \in G(z)$, the following inequality holds

$$U(g) \leq U(z).$$

As a result, for any solution z of system (3.5), the following inequality holds for any $t \geq 0$

$$\theta \int_0^t |x(s,k)|^2 ds \leq U(z(0,0)) + \theta\vartheta^2 \int_0^t |w(s,k)|^2 ds. \tag{3.16}$$

By using the fact that $(k_1^2 + k_2^2)^{\frac{1}{2}} \leq k_1 + k_2$ holds for any $k_1, k_2 \geq 0$, the inequality (3.15) can be derived from (3.16). □

3.5 Example

Consider the interconnected pendulum system which can be written into the form of system (3.1) with the following parameters

$$A_{11} := \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \ A_{12} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ B_1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ E_1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$A_{21} := A_{12}, \quad A_{22} := A_{11}, \quad B_2 := B_1, \quad E_2 = E_1.$$

The controller gains are chosen as:

$$K_1 := \begin{pmatrix} -6 & -8 \end{pmatrix}, \quad K_2 := \begin{pmatrix} -8 & -10 \end{pmatrix}.$$

We first consider the zero-order-hold case. Each local controller uses a zero-order-hold between two consecutive detecting instants. Choose $\varepsilon_1 = \varepsilon_2 = 0.001$, $\delta_1 = \delta_2 = 160$. Then we can design $\sigma_1 = \sigma_2 = 0.11$ and $T_1 = T_2 = 0.01$. The disturbance functions are set to be $w_1(t) = w_2 = 5e^{-2t}\sin(5t)$. The initial values are set to be $x_1(0) := (x_{11}(0), x_{12}(0))$ and $x_2(0) := (x_{21}(0), x_{22}(0))$ with $x_{11}(0) = x_{21}(0) = 0.1$ and $x_{12}(0) = x_{22}(0) = -0.1$. The quantizer parameters are set to be $\Delta = 0.5$ and $M = 30$. We can choose $L^1_{min} = L^2_{min} = 20$, $L_1 = L_2 = 22$, and $L^1_{max} = L^2_{max} = 30$. For comparison, we also consider a situation that all the subsystems adopt the model-based controllers and the parameters of the quantizers and triggering conditions are same as that of zero-order-hold cases.

The simulation time is 30 s. With the controller based on the zero-order-hold for each subsystem, Figs. 3.1, 3.2 and 3.3 are respectively the state trajectories, the release instants and release intervals for x_1 subsystem, and the quantization coefficient μ_1. Correspondingly, we draw the figures with the model-based controller

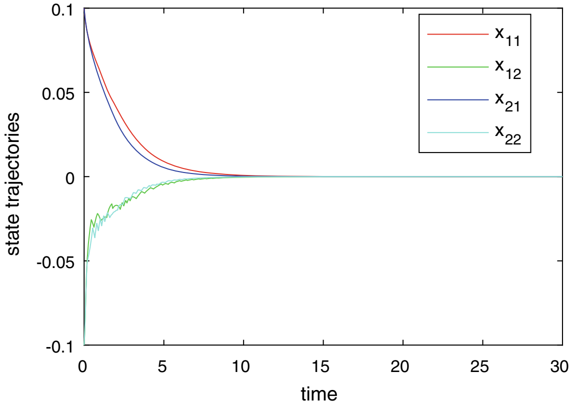


Fig. 3.1 State trajectories under zero-order-holds

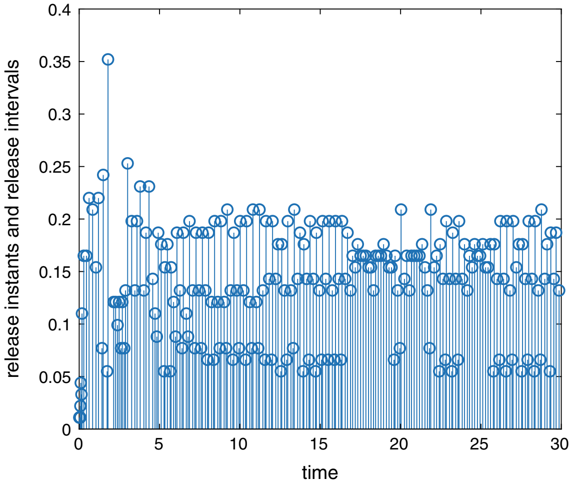


Fig. 3.2 Release instants and release intervals for x_1 subsystem under zero-order-holds

for each subsystem in Figs. 3.4, 3.5 and 3.6. With the controller based on zero-order-hold for each subsystem, the transmission number is 219 for x_1 subsystem and 235 for x_2 subsystem. With the model-based controller for each subsystem, the transmission number is 67 for x_1 subsystem and 132 for x_2 subsystem. The simulation results show that model-based controllers can generate less transmissions compared with the controllers based on the zero-order-hold.

Fig. 3.3 Quantization coefficient μ_1 for x_1 subsystem under zero-order-holds

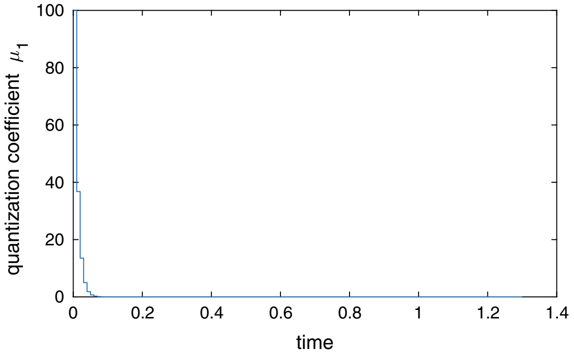


Fig. 3.4 State trajectories with model-based controllers

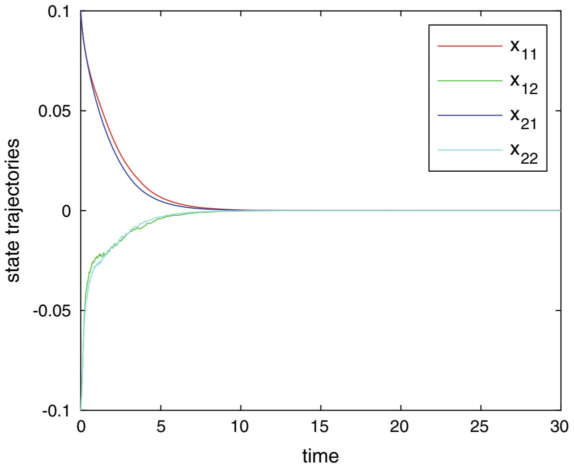
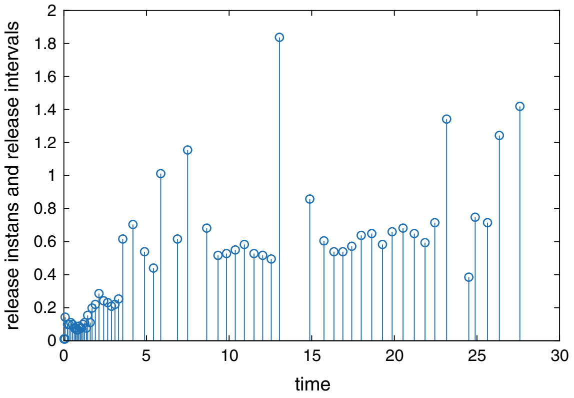


Fig. 3.5 Release instants and release intervals for x_1 subsystem with model-based controllers



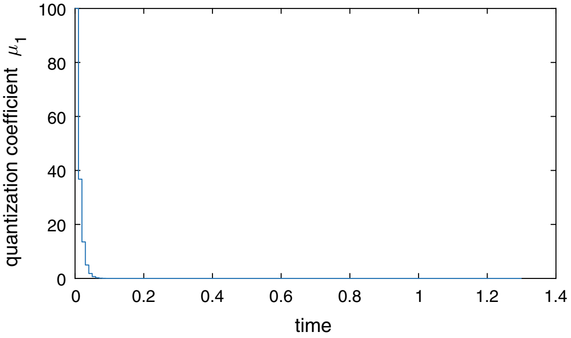


Fig. 3.6 Quantization coefficient μ_1 for x_1 subsystem with model-based controllers

3.6 Conclusions

A hybrid algorithm is proposed for decentralized periodic event-triggered control system with external disturbances. The proposed algorithm can ensure that the quantizer is not saturated. The entire system is modeled as a hybrid system. A stability condition is derived such that the hybrid system satisfies $\mathscr{L}_2$ gain performance in the presence of external disturbances.

Chapter 4
Event-Triggered Stabilization for Nonlinear Systems by Uniting the Local and Global Controller

4.1 Introduction

The main idea of event-triggered transmission is that a transmission happens only when it is necessary. In [80], the author proposes an event-triggered transmission strategy and the triggering condition is an inequality condition involving the system state and error variable. Then in [21], the author introduces an additional integrator dynamic and the transmissions are determined by the state of the integrator. Both of the authors in [80] and [21] prove that the Zeno phenomenon will not happen by showing that there is a minimal time between two adjacent transmission instants. For the case of output feedback, in [73] and [2], the authors impose a minimal time between two adjacent transmission instants. The Zeno phenomenon is avoided and asymptotic stability can be achieved in the absence of external disturbances. All of the works in [2, 21, 73, 80] require continuous checking of the triggering conditions. Differently, a periodic event-triggered transmission strategy does not check the triggering condition continuously, but checks it only at discrete detecting instants [28, 86, 107]. Linear systems are considered in [28, 107] where the former uses the hybrid system approaches and the latter uses a delay system approach. The more general case is the periodic event-triggered control for nonlinear systems. In [86], the authors model the nonlinear event-triggered control system as a hybrid system based on the framework in [7, 22, 23]. The hybrid system framework in [22] regards logic variables and timer variables as the system states and thus the researchers can construct Lyapunov functions containing these auxiliary variables [7, 9, 22, 23, 48]. The authors in [86] propose a novel Lyapunov function and stability conditions are derived based on Lyapunov functions.

For nonlinear event-triggered control systems, all the aforementioned literature use a single global controller which sometimes has a large control gain. It has been demonstrated in [68, 72, 84] that when the state is near the origin, a local controller may achieve better performance, for example, only a small control gain is needed. The small control gain may lead to less transmissions compared with the global controller. On the other hand, the local controller may not stabilize the nonlinear

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plant globally. This motivates us to consider a hybrid event-triggered transmission strategy. When the state is far from the origin, the global controller and a periodic event-triggered transmission strategy are used. In the event monitor at the plant side, there is a global norm observer [38, 75] that is used to estimate the norm of the system state. At the periodic detecting instants, if the estimated state norm is near the origin, the output and a logic variable are transmitted to the controller. Once receiving the information from the plant side, the global controller is replaced by a local controller. Similarly, there is also a local state norm observer to estimate the state norm of the local systems [6, 71]. At discrete detecting instants, if the estimated state norm becomes far from the origin, the plant output and a logic variable are transmitted to the controller. The local controller will be replaced by a global controller. The challenges are that a hybrid algorithm needs to be designed such that the entire system is globally asymptotically stable while local performance is preserved.

In this chapter, the main content can be summarized with the following points. The first one is that a hybrid periodic event-triggered transmission strategy is proposed for nonlinear systems. The hybrid transmission strategy consists of a global system and a local system. The global system has a global controller and a global periodic event-triggered transmission strategy. The local system has a local controller and a local periodic event-triggered transmission strategy. A hybrid algorithm is proposed to determine when a switch between the local system and global system happens. The second one is that the entire closed-loop system is modeled as a hybrid system which satisfies the hybrid basic conditions [22]. Global asymptotic stability is concluded for the resulting hybrid system which means that the proposed algorithm is also robustly stable with respect to small uncertainties. Two examples are given and the simulations show that the proposed hybrid algorithm reduces the network transmissions significantly.

4.2 Motivation Example and Problem Description

Consider the following scalar system

$$\dot{x} = -x^3 + 6x^2 + u \tag{4.1}$$

where u is the control input. A network-based static controller is given as follows

$$u = -K\hat{x} \tag{4.2}$$

where $\hat{x}$ is the most recently transmitted plant state. Because there is a quadratic function $6x^2$ in system (4.1), we choose a relatively large control gain to stabilize system (4.1) globally. But when the state variable x is near the origin, the high order terms $-x^3$ and $6x^2$ can be neglected and thus the control gain K can be chosen as a small number. Specifically, let $e := \hat{x} - x$. Then we have

$$\dot{x} = -x^3 + 6x^2 - Kx - Ke.$$

Choose a function $V(x) := \frac{1}{2}x^2$. We can derive

$$\begin{aligned}\dot{V} &= -x^4 + 6x^3 - Kx^2 - Kxe \\ &\leq 9x^2 - Kx^2 - Kxe.\end{aligned}$$

K can be chosen larger than 9. When the state is near the origin, we can derive according to the linearized model $\dot{V} = -Kx^2 - Kxe$. In this case, we can choose a small gain, for example $\frac{1}{2}$. Because the transmission of output is determined by an event-triggered strategy, we choose a hybrid transmission strategy. When the state is far from the origin, we use a global controller and a corresponding event-triggered transmission strategy such that the origin is globally asymptotically stable. When the state is near the origin, the event monitor sends a logic variable to the controller and the global controller is replaced by a local one. On the other hand, when the state becomes far from the origin, the local controller is replaced to the global one. Moreover, different transmission strategies are adopted along with different controllers. Simulation shows that a small control gain leads to less transmissions under the same triggering condition, for example, $|e| \geq 0.3|x|$. This motivates the hybrid event-triggered controller design.

4.3 Hybrid Event-Triggered Stabilization

Consider the following nonlinear plant

$$\dot{x} = f_p(x, u), \quad y = h_p(x) \tag{4.3}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input and $y \in \mathbb{R}^{n_y}$ is the output. The functions f_p and h_p are continuous with $h_p(0) = 0$. We consider the case that the output of system (4.3) is transmitted via the network to the controller. The transmission times are determined by an event-triggered transmission strategy. We use the combination of a local event-triggered controller and a global event-triggered controller. The global controller has the following form

$$u = k_1(\hat{y})$$

where $\hat{y}$ is the most recently transmitted plant output and k_1 is a continuous function with $k_1(0) = 0$. When the global controller is used, the transmission times are generated by a periodic event-triggered transmission strategy. More specifically, let $\{t_k^1\}_{k=1}^{\infty}$ be a periodic time sequence with $t_{k+1}^1 - t_k^1 = T_1$ where $T_1 > 0$ is a detecting period. At each time t_k^1, the following triggering condition will be checked

$$W(e) \geq \widetilde{\sigma}_1(y) \tag{4.4}$$

where $e := \hat{y} - y \in \mathbb{R}^{n_e}$. $W : \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ and $\widetilde{\sigma}_1 : \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0}$ are continuous functions to be designed. Once the condition (4.4) is satisfied at the detecting times, the output is sampled and transmitted via the network to the global controller. The entire closed-loop system with the global event-triggered controller can be formulated as the following hybrid system [22]

$$\begin{cases} \dot{x} = f_1(x, e, 1) & \\ \dot{e} = f_2(x, e, 1) & \tau \in [0, T_1] \\ \dot{\tau} = 1 & \\ x^+ = x & \\ e^+ \in \tilde{g}_1(x, e, \tau) & \tau = T_1 \\ \tau^+ = 0 & \end{cases} \tag{4.5}$$

where $f_1(x, e, 1) := f_p(x, k_1(h_p(x) + e))$, $f_2(x, e, 1) := -\frac{\partial h_p(x)}{\partial x} f_p(x, k_1(h_p(x) + e))$ and

$$\tilde{g}_1(x, e, \tau) := \begin{cases} e, & W(e) < \widetilde{\sigma}_1(y) \\ 0, & W(e) > \widetilde{\sigma}_1(y) \\ \{e, 0\}, & W(e) = \widetilde{\sigma}_1(y). \end{cases}$$

When the state x is near the origin, we use a local controller

$$u = k_0(\hat{y})$$

and a local periodic event-triggered transmission strategy. k_0 is a continuous function with $k_0(0) = 0$. For the local periodic event-triggered transmission strategy, the detecting period is $T_0 > 0$ and the triggering condition is as follows

$$W(e) \geq \widetilde{\sigma}_0(y).$$

Then the entire closed-loop system with the local event-triggered controller can be formulated into the following hybrid system

$$\begin{cases} \dot{x} = f_1(x, e, 0) & \\ \dot{e} = f_2(x, e, 0) & \tau \in [0, T_0] \\ \dot{\tau} = 1 & \\ x^+ = x & \\ e^+ \in \tilde{g}_0(x, e, \tau) & \tau = T_0 \\ \tau^+ = 0 & \end{cases} \tag{4.6}$$

where $f_1(x, e, 0) := f_p(x, k_0(h_p(x) + e)), f_2(x, e, 0) := -\frac{\partial h_p(x)}{\partial x} f_p(x, k_0(h_p(x) + e))$ and

$$\tilde{g}_0(x, e, \tau) := \begin{cases} e, & W(e) < \widetilde{\sigma}_0(y) \\ 0, & W(e) > \widetilde{\sigma}_0(y) \\ \{e, 0\}, & W(e) = \widetilde{\sigma}_0(y). \end{cases}$$

As explained in the motivation example, the global event-triggered controller can stabilize the plant globally which means that the set $\mathscr{W}_1 := \{0\} \times \{0\} \times [0, T_1]$ is globally attractive for system (4.5). However, the global controller may need a large control gain. In contrast, the local event-triggered controller may need a small control gain but may only stabilize the plant locally which means that the set $\mathscr{W}_0 := \{0\} \times \{0\} \times [0, T_0]$ is locally asymptotically stable for system (4.6). Our aim is to seek a hybrid algorithm which combines the local event-triggered controller and global event-triggered controller. When the state is far away from the origin, a global controller is used and when the state enters near the origin, a local controller is used. We want to explore a hybrid algorithm such that the entire system is globally asymptotically stable while the local performance is still preserved. Moreover, the hybrid algorithm should have certain robustness against the small measurement errors and uncertainties. Let $q \in \{0, 1\}$ be a logic variable. $q = 1$ means that a global controller is used and $q = 0$ means that a local controller is used.

Next, a formal assumption about the global controller and local controller is formulated as follows.

Assumption 4.1 The set $\mathscr{W}_1$ is globally attractive for system (4.5). The set $\mathscr{W}_0$ is locally asymptotically stable for system (4.6). Moreover, the solution of the following ordinary differential equation does not blow up in finite time

$$\begin{cases} \dot{x} = f_1(x, e, 0) \\ \dot{e} = f_2(x, e, 0). \end{cases} \tag{4.7}$$

Remark 4.1 The assumption that the solution of system (4.7) does not blow up in finite time is equivalent to that any solution is forward complete, which are mild constraints. Note that we have not presented the explicit conditions such that Assumption 4.1 holds because the design of periodic event-triggered controller for nonlinear systems has been well developed in the literature [86]. We do not introduce the conditions here to avoid redundancies. It is also interesting to consider the case that each controller uses a static event-triggered transmission strategy or a dynamic event-triggered transmission strategy. In the example section, we will present the detailed design procedures.

The global convergence guarantees that the state variables x and e will converge to the origin. Because only the output y is accessible, we need to construct a norm observer to estimate the norm of the state x so that we know when to use a local controller. Similarly, we also need a norm observer for the local system so that we

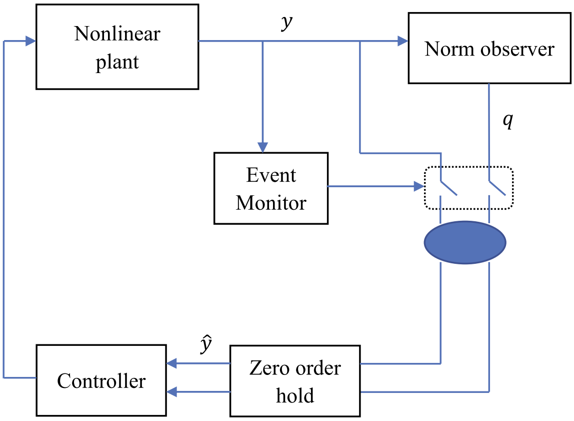


Fig. 4.1 Structure of the hybrid event-triggered control system

know when to use a global controller. The norm observers can be implemented in the event monitor as illustrated in Fig. 4.1. The following assumptions are needed.

Assumption 4.2 Suppose that there exist two continuous functions $U_0 : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times [0, T_0] \to \mathbb{R}_{\geq 0}$ and $V_0 : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, functions $\underline{\alpha}_{U_0}, \overline{\alpha}_{U_0}, \underline{\alpha}_V, \overline{\alpha}_V \in \mathscr{K}_\infty$, functions $\beta_i \in \mathscr{K}\mathscr{L} (i = 0, 1)$, continuous positive definite functions ρ_i (i=0,1), numbers $0 < \varepsilon_{0a} < \varepsilon_{0b}$ and $0 < \varepsilon_{1a} < \varepsilon_{1b}$ such that

(1) $\underline{\alpha}_{U_0}(|(x, e, \tau)|_{\mathscr{W}_0}) \leq U_0(x, e, \tau) \leq \overline{\alpha}_{U_0}(|(x, e, \tau)|_{\mathscr{W}_0})$ and $\underline{\alpha}_V(|x|) \leq V_0(x) \leq \overline{\alpha}_V(|x|)$ for all $(x, e, \tau) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times [0, T_0]$,

(2) The attraction domain of $\mathscr{W}_0$ for system (4.6) contains the set $\{(x, e, \tau) | \tau \in [0, T_0], U_0(x, e, \tau) \leq \varepsilon_{1a}\}$,

(3) the following global norm observer for system (4.5)

$$\dot{z}_1 = -z_1 + \rho_1(u, y)$$

is such that $V_0(x(t, j)) \leq z_1(t, j) + \beta_1(|(x_0^1, z_0^1)|, t)$ for any initial value (x_0^1, z_0^1) of (x, z_1),

(4) the following local norm observer of system (4.6)

$$\dot{z}_0 = -z_0 + \rho_0(e, y)$$

is such that $U_0(x(t, j), e(t, j), \tau(t, j)) \leq z_0(t, j) + \beta_0(|(x_0^0, z_0^0)|, t)$ for any initial value (x_0^0, z_0^0) of (x, z_0),

(5) any solution (x, e, τ) of system (4.6) starting from $\{(x_0, 0, 0) : V_0(x_0) \leq \varepsilon_{1b}\}$ satisfies $\rho_0(e(t, j), y(t, j)) \leq \varepsilon_{0a}$ for all $(t, j) \in \mathrm{dom}(x, e, \tau)$.

Remark 4.2 For continuous-time systems, the existence of a norm observer can be deduced from input-output-to-state stability [38, 75]. Let us recall input-output-to-state stability for the nonlinear plant (4.3). System (4.3) is said to be input-output-to-state stable (IOSS) if there exist two functions $\tilde{\beta} \in \mathcal{KL}$ and $\tilde{\gamma} \in \mathcal{K}_\infty$ such that any solution x of system (4.3) satisfies the following inequality

$$|x(t)| \leq \max\{\tilde{\beta}(|x_0|, t), \tilde{\gamma}(\sup_{s\in[0,t]} |y(s)|), \tilde{\gamma}(\sup_{s\in[0,t]} |u(s)|)\}.$$

IOSS implies the existence of a continuously differentiable function $\tilde{V} : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ such that

- There exist functions $\nu_1, \nu_2 \in \mathcal{K}_\infty$ satisfying $\nu_1(|x|) \leq \tilde{V}(x) \leq \nu_2(|x|)$ for all $x \in \mathbb{R}^{n_x}$,
- There exist functions $\tilde{\rho}_1^1, \tilde{\rho}_2^1 \in \mathcal{K}$ such that

$$\nabla\tilde{V}(x)f(x,u) \leq -\tilde{V}(x) + \tilde{\rho}_1^1(|u|) + \tilde{\rho}_2^1(|y|).$$

Then $\rho_1(u, y)$ can be taken as $\rho_1(u, y) := \tilde{\rho}_1(|u|) + \tilde{\rho}_2(|y|)$. Note that the system (4.5) satisfies the dwell-time condition. Therefore, a global norm observer can be constructed as given in Assumption 4.2. A norm observer for hybrid systems can be deduced from output-to-state stability. Therefore, the main assumptions are an assumption on IOSS for the continuous plant and an assumption on output-to-state stability for hybrid system (4.6). These results have been well developed in the literature [6, 38, 71, 75].

Now we are ready to propose the following hybrid event-triggered control algorithm.

Hybrid algorithm:

(1) When $q = 1$, the global event-triggered controller is used. The state variable (x, e, τ) is governed by the hybrid system (4.5). In the meantime, at some periodic detecting time, if $z_1 < \varepsilon_{0b}$, then the logic variable q is reset to 0. The logic variable and the plant output y are sent to the controller. Once receiving the information from the plant side, the global controller is replaced by a local event-triggered controller and $\hat{y}$ is updated as the received plant output y.

(2) When $q = 0$ and $z_0 \leq \varepsilon_{0a}$, the local event-triggered controller is used. The state variable (x, e, τ) is governed by the hybrid system (4.6). In the meantime, at some periodic detecting time, if $z_0 > \varepsilon_{0a}$, then the logic variable is reset to 1. The logic signal and the plant output y are sent to the controller. The local controller is replaced by the global event-triggered controller and $\hat{y}$ is updated as the received output y. z_0 is also reset to zero. △

Remark 4.3 If the plant state is accessible, we do not need the norm observers and the hybrid algorithm can be simplified. The jump condition from the global controller to local controller is $V_0(x) \leq \varepsilon_{0b}$ and the jump condition from the local controller to global controller is $U_0(x, e, \tau) > \varepsilon_{0a}$. Besides, $\{x|V_0(x) \leq \varepsilon_{0b}\}$ is a subset of $\{x|U_0(x, 0, 0) \leq \varepsilon_{1a}\}$ and any solution of system (4.6) starting from $\{(x_0, 0, 0)|V_0(x_0) \leq \varepsilon_{0b}\}$ satisfies $U_0(x, e, \tau) \leq \varepsilon_{0a}$. As a result, when the global system is replaced by the local system, the state is within the attraction domain of the local system.

Because the global controller stabilizes the plant globally, the global controller can be replaced by the local controller after sufficiently long time.

Let $\chi := (x, e, z_1, z_0, \tau, q)$. The entire closed-loop system can be formulated into the following hybrid system

$$\begin{cases} \dot{\chi} = F(\chi), & \chi \in C \\ \chi^+ \in G(\chi), & \chi \in D. \end{cases} \tag{4.8}$$

The flow dynamics are specified as follows

$$\begin{cases} \dot{x} = f_1(x, e, q) \\ \dot{e} = f_2(x, e, q) \\ \dot{z}_1 = -z_1 + \rho_1(k_c(h_p(x) + e), h_p(x)) \\ \dot{z}_0 = (1 - q)(-z_0 + \rho_0(e, h_p(x))) \\ \dot{\tau} = 1 \\ \dot{q} = 0. \end{cases}$$

The flow set is $C := C_1 \cup C_2$ where

$$\begin{aligned} C_1 &:= \{\chi : q = 1,\ \tau \in [0, T_1],\ z_0 = 0,\ z_1 \geq 0\} \\ C_2 &:= \{\chi : q = 0,\ \tau \in [0, T_0],\ z_1 \geq 0,\ z_0 \geq 0\}. \end{aligned}$$

The jump set is $D := \cup_{i=1}^{6} D_i$ where

$$\begin{aligned} D_1 &:= \{\chi : q = 1,\ \tau = T_1,\ z_1 < \varepsilon_{0b},\ z_0 = 0\} \\ D_2 &:= \{\chi : q = 1,\ \tau = T_1,\ z_1 > \varepsilon_{0b},\ z_0 = 0\} \\ D_3 &:= \{\chi : q = 1,\ \tau = T_1,\ z_1 = \varepsilon_{0b},\ z_0 = 0\} \\ D_4 &:= \{\chi : q = 0,\ \tau = T_0,\ z_1 \geq 0,\ z_0 > \varepsilon_{0a}\} \\ D_5 &:= \{\chi : q = 0,\ \tau = T_0,\ z_1 \geq 0,\ z_0 < \varepsilon_{0a}\} \\ D_6 &:= \{\chi : q = 0,\ \tau = T_0,\ z_1 \geq 0,\ z_0 = \varepsilon_{0a}\}. \end{aligned}$$

The jump map is

$$G(\chi) := \begin{cases} (x, 0, z_1, z_0, 0, 0), & \chi \in D_1 \\ G_1(\chi), & \chi \in D_2 \\ (x, 0, z_1, z_0, 0, 0) \cup G_1(\chi), & \chi \in D_3 \\ (x, 0, z_1, 0, 0, 1), & \chi \in D_4 \\ G_2(\chi), & \chi \in D_5 \\ (x, 0, z_1, 0, 0, 1) \cup G_2(\chi), & \chi \in D_6 \end{cases}$$

where

$$G_1(\chi) := \begin{cases} (x, 0, z_1, z_0, 0, 1), & W(e) > \widetilde{\sigma}_1(x) \\ (x, e, z_1, z_0, 0, 1), & W(e) < \widetilde{\sigma}_1(x) \\ (x, 0, z_1, z_0, 0, 1) \cup (x, e, z_1, z_0, 0, 1), & W(e) = \widetilde{\sigma}_1(x) \end{cases}$$

and

$$G_2(\chi) := \begin{cases} (x, 0, z_1, z_0, 0, 0), & W(e) > \widetilde{\sigma}_0(y) \\ (x, e, z_1, z_0, 0, 0), & W(e) < \widetilde{\sigma}_0(y) \\ (x, 0, z_1, z_0, 0, 0) \cup (x, e, z_1, z_0, 0, 0), & W(e) = \widetilde{\sigma}_0(y). \end{cases}$$

From the flow set and jump set, we can see that at the discrete detecting instants, the event monitor at the plant side checks not only the triggering condition but also the switching condition to determine whether a controller switching is necessary. We will show under the proposed hybrid algorithm, no matter what the initial logic variable q is, q will eventually jump to 0 after finite jumps. Moreover, the state will converge to the equilibrium set.

The flow set and the jump set are closed subsets of $\mathbb{R}^{n_\chi}$. Because the functions $f_p, h_p, k_1, k_0, \rho_0, \rho_1$ are continuous, the functions f_1 and f_2 are continuous with respect to the variables x, e, q relative to the flow set. Then the flow map F is continuous relative to the flow set. The jump map is continuous at the points belonging to D_1, D_4 and outer semicontinuous at the points belonging to the sets D_2, D_3, D_5. Then the jump map G is outer semicontinuous with respect to the jump set. Therefore, the data of hybrid system (4.8) satisfies the hybrid basic conditions [22]. We are ready to present one of the main theorems. The proof is inspired by that of [68].

Theorem 4.1 *Consider system (4.8) and assume Assumptions 4.1, 4.2 hold. Then the set* $\mathscr{W} := \{0\} \times \{0\} \times \{0\} \times \{0\} \times [0, T_0] \times \{0\}$ *is globally asymptotically stable for system (4.8).* △

Proof of Theorem 4.1 To show global asymptotic stability, it is enough to show the set $\mathscr{W}$ is locally stable and globally attractive. We firstly show local stability. Let ϵ be an arbitrary number with $0 < \epsilon < \varepsilon_{0a}$. Because the functions $\rho_0, \rho_1, h_p, k_0, k_1$ are continuous and vanish at zero, there exists a $\delta_2 > 0$ such that $\rho_1(k_1(e + h_p(x)), h_p(x)), \rho_0(e, h_p(x)) \le \frac{\epsilon}{8}$ for all $|(x, e)| \le \delta_2$. From the stability assumptions in Assumption 4.1, there exists a $\delta_1 > 0$ such that any solution (x, e, τ) of system (4.6) with initial value $|(x_0, e_0, \tau_0)|_{\mathscr{W}_0} \le \delta_1$ satisfies $|(x(t, j), e(t, j), \tau(t, j))|_{\mathscr{W}_0} \le \min\{\frac{\epsilon}{2}, \delta_2\}$. The initial values of z_0 and z_1 can be chosen less than $\frac{\epsilon}{8}$. Then z_0 and z_1 are less than $\frac{\epsilon}{4}$. Therefore, we can choose $\delta < \min\{\delta_1, 1\}$. Any solution χ with initial value $|\chi_0|_{\mathscr{W}} \le \delta$ satisfies $|\chi(t, j)|_{\mathscr{W}} \le \epsilon$ for any $(t, j) \in \operatorname{dom}\chi$.

Now we prove global attractivity. The proving procedures rely on the following two claims.

Claim 1 For any solution χ of system (4.8), if there exists a hybrid time $(t_k, j_k) \in \operatorname{dom}\chi$ such that $q(t, j) = 0$ for all $(t, j) \in \operatorname{dom}\chi$ with $t + j \geq t_k + j_k$, then $\lim_{t+j\to\infty} |\chi(t, j)|_{\mathscr{W}} = 0$.

Proof Suppose that there exists a hybrid time $(t_k, j_k) \in \operatorname{dom}\chi$ such that $q(t, j) = 0$ for all $(t, j) \in \operatorname{dom}\chi$ with $t + j \geq t_k + j_k$. Then for $\chi(t, j) \in D$ and $t + j \geq t_k + j_k$, $z_0(t, j) \leq \epsilon_{0a}$. For sufficiently large $t + j$ with $(t, j) \in \operatorname{dom}\chi$, β_0 is sufficiently small and thus $U_0(x(t, j), e(t, j), \tau(t, j)) \leq \epsilon_{1a}$ according to condition (4) of Assumption 4.2. Then according to condition (2) of Assumption 4.2, $\chi(t, j)$ approaches the set $\mathscr{W}$ as $t + j$ approaches ∞.

Claim 2 For any solution χ of system (4.8), there does not exist a nondecreasing sequence of hybrid times $(t_k, j_k) \in \operatorname{dom}\chi$ such that

$$q(t_{2k}, j_{2k}) = 0, \quad q(t_{2k+1}, j_{2k+1}) = 1. \tag{4.9}$$

Proof We prove Claim 2 by contradiction. Suppose that there exists a sequence satisfying (4.9). Without loss of generality, we can assume that between (t_{2k}, j_{2k}) and $(t_{2k+1}, j_{2k+1} - 1)$, $q = 0$ and between (t_{2k+1}, j_{2k+1}) and $(t_{2k+2}, j_{2k+2} - 1)$, $q = 1$. The logic variable q is updated as 1 from 0 at the instant $(t_{2k+1}, j_{2k+1} - 1)$. It is updated as 0 from 1 at the instant $(t_{2k+2}, t_{2k+2} - 1)$. From the jump set, we know $z_1(t_{2k+2}, t_{2k+2} - 1) \leq \epsilon_{0b}$. Because each two adjacent jumps are separated by a positive number, for sufficiently large $t_{2k+2} + j_{2k+2}$, we have $V_0(x(t_{2k+2}, t_{2k+2} - 1)) \leq \epsilon_{1b}$ by using condition (3) of Assumption 4.2. Besides, $e(t_{2k+2}, j_{2k+2}) = 0$ based on the jump map. Combining conditions (4), (5) of Assumption 4.2 and $z_0(t_{2k+2}, j_{2k+2}) = 0$, we have $z_0(t, j) \leq \epsilon_{0a}$ for all $(t, j) \in \operatorname{dom}\chi$ with $t + j \geq t_{2k+2} + j_{2k+2}$. Therefore, $q(t, j) = 0$ for all $(t, j) \in \operatorname{dom}\chi$ with $t + j \geq t_{2k+2} + j_{2k+2}$ which is a contradiction.

Combining Claim 1 and Claim 2, we can conclude that the set $\mathscr{W}$ is globally attractive and thus globally asymptotically stable. The proof is completed. □

We have shown that the proposed hybrid event-triggered controller stabilizes the nonlinear plant globally. In practical applications, there exist various kinds of uncertainties such as measurement errors and model uncertainties. Next, we will show that the proposed hybrid algorithm is also robust to such uncertainties. Consider the perturbed hybrid system of system (4.8) as follows

$$\begin{cases} \dot{\chi} \in F_\delta(\chi), & \chi \in C_\delta \\ \chi^+ \in G_\delta(\chi), & \chi \in D_\delta \end{cases} \tag{4.10}$$

where the data are specified as follows

$$
\begin{aligned}
F_\delta(\chi) &:= \overline{\mathrm{co}}F(\chi + \delta\mathbb{B}) + \delta\mathbb{B} \\
G_\delta(\chi) &:= \{v : v \in g + \delta\mathbb{B},\ g \in G(\chi + \delta\mathbb{B})\} \\
C_\delta(\chi) &:= \{\chi : (\chi + \delta\mathbb{B}) \cap C \neq \emptyset\} \\
D_\delta(\chi) &:= \{\chi : (\chi + \delta\mathbb{B}) \cap D \neq \emptyset\}.
\end{aligned}
$$

The following robustness result can be concluded.

Theorem 4.2 *Consider system (4.8) and assume Assumptions 4.1, 4.2 hold. Then the set $\mathscr{W}$ is robustly stable for system (4.8). Specifically, there exists a function $\beta \in \mathscr{K}\mathscr{L}$ such that for any compact set $\mathscr{O}$ and number $\epsilon > 0$, there exists a $\delta > 0$ such that any solution χ starting from $\mathscr{O}$ of the perturbed system satisfies $|\chi(t, j)|_{\mathscr{W}} \leq \beta(|\chi(0, 0)|_{\mathscr{W}}, t + j) + \epsilon$.* △

Proof Because the set $\mathscr{W}$ is compact and the data of system (4.8) satisfies the hybrid basic conditions. The proof is completed by applying Lemma 7.20 of [22].

4.4 Example Studies

In this section, we consider two examples to illustrate the improvement over the existing literature. One example considers the state feedback control and the other one considers the output feedback control. We will compare the hybrid event-triggered controller in this chapter with the global periodic event-triggered controller [86].

Example 1 Consider the following system from [4]

$$\dot{x} = x^2 - x^3 + u. \tag{4.11}$$

The existing literature uses the global controller $u = -2\hat{x}$. Differently, near the origin, we will use the following local controller $u = -0.1\hat{x}$. It can be verified by simulation that the local controller cannot stabilize the nonlinear plant globally because of the small control gain.

Global event-triggered stabilization:

The function $f_2(x, e, 0)$ satisfies $|f_2(x, e, 0)| \leq L_1|e| + H_1(x)$ with $H_1(x) := |x^2 - x^3 - 2x|$ and $L_1 = 2$. For the global periodic event-triggered control, choose the function $V_1(x) := \frac{d_1}{2}x^2 + \frac{d_2}{4}x^4$ for the x subsystem. Then we have

$$
\begin{aligned}
\nabla V_1(x) f_1(x, e) = (x^3 - x^4 - 2x^2 - 2ex)d_1 \\
+ (x^5 - x^6 - 2x^4 - 2x^3e)d_2 \\
+ 5x^4 + x^6 + 4x^2 - 2x^5 - 4x^3 - H^2(x).
\end{aligned}
$$

Choose $d_1 = 8$ and $d_2 = 2$. Then we can obtain

$$\begin{aligned}\nabla V_1(x) f_1(x, e) &= -x^6 - 3x^4 - 4x^3 - 8x^2 - 16ex - 4x^3 e - H^2(x)\\ &\leq -x^4 - 4x^2 - e^2 + \gamma_1^2 e^2 - H^2(x)\end{aligned}$$

with $\gamma_1 = 6.1$.

Consider the Lyapunov function $U_1(x, e, \tau) = V_1(x) + \max\{\frac{2}{\gamma_1}|x|^2, \gamma_1\phi_1(\tau)W^2(e)\}$ with $W(e) = |e|$. The function ϕ_1 satisfies the following differential equation

$$\dot{\phi}_1(\tau) = -2L_1\phi_1(\tau) - \gamma_1\phi_1^2(\tau) - \gamma_1, \quad \tau \in [0, T_1].$$

Choose $\phi_1(0) = 2$ and $\phi_1(T_1) = 0.5$. Then T_1 can be solved to be 0.08. We compute the Clarke's general derivative U_1° of U_1 and the computation can be divided into three cases. One case is $\frac{2}{\gamma_1}|x|^2 > \gamma_1\phi_1 W^2(e)$, the second case is $\frac{2}{\gamma_1}|x|^2 < \gamma_1\phi_1 W^2(e)$ and the third case is $\frac{2}{\gamma_1}|x|^2 = \gamma_1\phi_1 W^2(e)$. For all the cases, we can derive $U_1^\circ(x, e, \tau) \leq -\alpha(|(x, e, \tau)|_{\mathscr{W}_1})$ for some function $\alpha \in \mathscr{K}_\infty$. In the jump set, when $W(e) > \widetilde{\sigma}_1(y)$, e is reset as zero and thus the function U_1 does not increase. Let $\widetilde{\sigma}(y) := 0.1|x|$. When $W(e) < \widetilde{\sigma}_1(y)$, e does not change and thus $U_1(x^+, e^+, \tau^+) - U_1(x, , e, \tau) \leq \max\{\frac{2}{\gamma_1}|x|^2, \gamma_1\phi_1(0)W^2(e)\} - \frac{2}{\gamma_1}|x|^2$ because $W(e) < \widetilde{\sigma}_1(y)$ implies $W^2(e) < \frac{1}{\gamma_1^2}|x|^2$. The function U_1 decreases during continuous evolution and does not increase during jumps and the jumps are separated by a constant time. Therefore, $\mathscr{W}_1$ is globally asymptotically stable.

Hybrid event-triggered stabilization:

For the local event-triggered control system, we have $|f_2(x, e, 0)| \leq L_0|e| + H_0(x)$ where $L_0 = 0.1$ and $H_0(x) := |x^2 - x^3 - 0.1x|$.

For the local system, choose the function $U_0(\chi) := V_0(x) + \max\{b|x|^2, \gamma_0\phi_0(\tau)W^2(e)\}$ where $V_0(x) := 4x^2$ and $b = 0.1$. When $b|x|^2 < \gamma_0\phi_0(\tau)W^2(e)$, we can derive

$$\begin{aligned}U_0^\circ(x, e, \tau) &= -0.8x^2 - 0.8ex + (8x^3 - 8x^4) + \gamma_0\dot{\phi}_0 W^2(e)\\ &\quad + 2\gamma_0\phi_0 W(e)(L_0 W(e) + H_0(x))\end{aligned}$$

where ϕ_0 satisfies $\dot{\phi}_0 = -2L_0\phi_0 - 2\phi_0^2 - 2$.

When $b|x|^2 > \gamma_0\phi_0(\tau)W^2(e)$, we can derive

$$\begin{aligned}U_0^\circ &= 8x(x^2 - x^3 - 0.1x - 0.1e) + 2bx(x^2 - x^3 - 0.1x - 0.1e)\\ &= -0.8x^2 - 0.8ex + 8x^3 - 8x^4 + 2bx(x^2 - x^3 - 0.1x - 0.1e).\end{aligned}$$

To compare fairly, choose $T_0 = T_1$ and $\widetilde{\sigma}_0(y) = 0.1|x|$. Then $\{(x, e, \tau)|U_0(x, e, \tau)\} \leq 4 \times 10^{-4}$ is an attraction domain because in this set, we have $|x| \leq 0.01$ which leads to $U_0^\circ \leq -\alpha_0(|(x, , e, \tau)|_{\mathscr{W}_0})$ for some $\alpha_0 \in \mathscr{K}_\infty$ and $U_0(x^+, e^+, \tau^+) \leq U_0(x, e, \tau)$. We can choose $\varepsilon_{0b} = 2 \times 10^{-4}$ and $\varepsilon_{0a} = 4 \times 10^{-4}$.

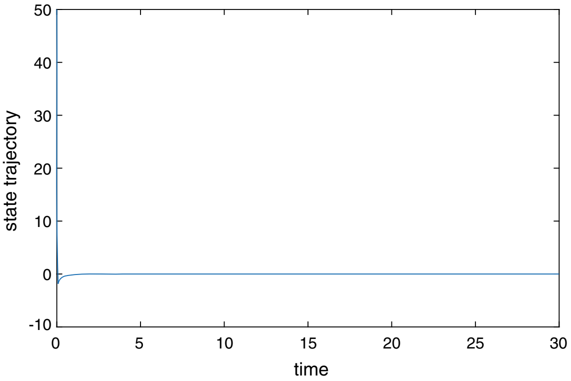


Fig. 4.2 State trajectory with global controller

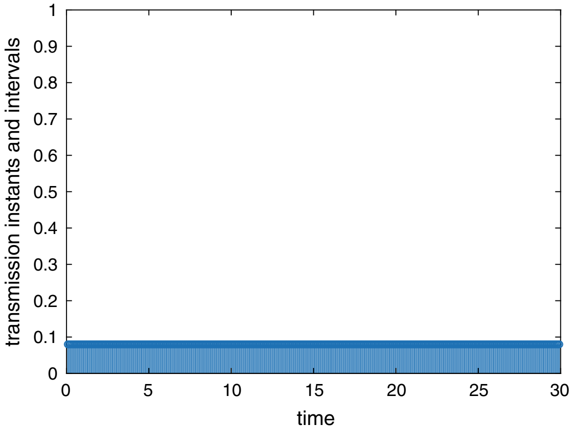


Fig. 4.3 Transmission instants and intervals with global controller

The initial state for x is 50. The simulation time is 30 s. The state trajectories are plotted in Figs. 4.2 and 4.4 for different controllers. The transmission instants and transmission intervals are plotted in Figs. 4.3 and 4.5. q is plotted in Fig. 4.6. It can be observed from Figs. 4.3, 4.5 and 4.6 that the logic variable finally jumps to 0 and the transmission intervals of hybrid controller are significantly larger than that of the global controller when the state enters near the origin. This shows that the proposed algorithm reduces the transmission number heavily.

Example 2 Consider the following system where only part of the state is available

$$\begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 + u \\ \dot{x}_2 = -x_2 + x_1^2 \\ \ \ y = x_1. \end{cases} \tag{4.12}$$

The global controller is designed as $u = -4\hat{x}_1$. Near the origin, the local controller is designed as $u = -0.2\hat{x}_1$. The function f_2 satisfies $|f_2(x, e, 1)| \le L_1|e| + H_1(x)$

Fig. 4.4 State trajectory with hybrid controller

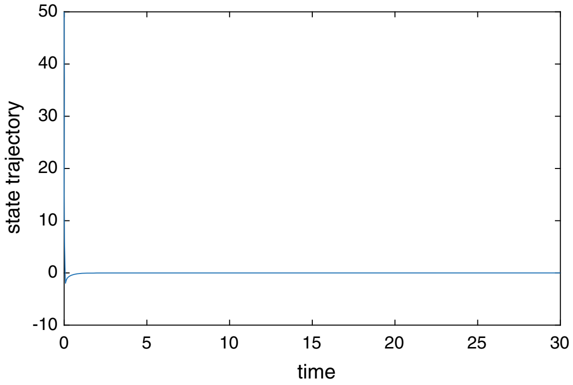


Fig. 4.5 Transmission instants and intervals with hybrid controller

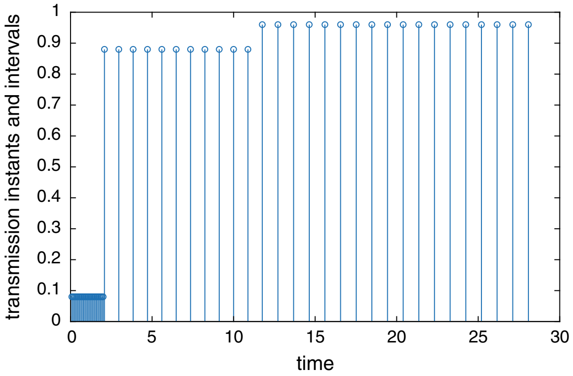
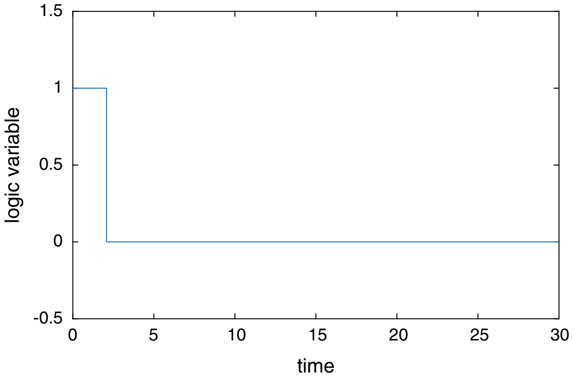


Fig. 4.6 Logic variable with hybrid controller



where $L_1 = 4$ and $H_1(x) = |x_1^2 - x_1^3 + x_2 - 4x_1|$. We firstly design the parameters for the global systems. Choose the function $V_1(x) := 4x_1^2 + 2x_1^4 + 2x_2^2$. We can compute that

$$\begin{aligned}
\frac{\partial V_1(x)}{\partial x} f_1(x,e) &= 8x_1(x_1^2 - x_1^3 + x_2 - 4x_1 - 4e) \\
&\quad + 8x_1^3(x_1^2 - x_1^3 + x_2 - 4x_1 - 4e) + 4x_2(-x_2 + x_1^2) \\
&\quad + x_1^6 - 2x_1^5 + 9x_1^4 - 2x_2x_1^3 - 8x_1^3 + 2x_1^2x_2 + 16x_1^2 \\
&\quad + x_2^2 - 8x_1x_2 - H^2(x) \\
&= -7x_1^6 + 6x_1^5 - 31x_1^4 + 6x_2x_1^3 + 6x_1^2x_2 - 16x_1^2 - 3x_2^2 \\
&\quad - 32x_1e - 32x_1^3e - H^2(x) \\
&\le -\frac{3}{2}x_1^6 - 4x_1^4 - 16x_1^2 - \frac{1}{2}x_2^2 - 32x_1e - 32x_1^3e - H^2(x) \\
&\le -4x_1^4 - 12x_1^2 - \frac{1}{2}x_2^2 + \gamma^2W^2(e) - H^2(x) \qquad (4.13)
\end{aligned}$$

where $\gamma_1 = 16$. Choose the function $U_1(x,e,\tau) := V_1(x) + \max\{\frac{a}{\gamma_1}|x_1|^2, \gamma_1\phi_1(\tau)W^2(e)\}$ where ϕ_1 satisfies $\dot{\phi}_1 = -2L_1\phi_1 - \gamma_1\phi_1^2 - \gamma_1$. Let $\phi_1(0) = 2$ and $\phi_1(T_1) = 0.5$ which leads to $T_1 = 0.03$. Choose $a = 4$ and $\widetilde{\sigma}_1(y) := \frac{1.4}{16}|x_1|$. We can verify $U_1^\circ \le -\alpha(|(x,e,\tau)|_{\mathcal{W}_1})$ for some function $\alpha \in \mathcal{K}_\infty$ and $U_1(x^+, e^+, \tau^+) \le U_1(x,e,\tau)$ for $\tau = T_1$. Because the jumps are separated by a constant time, the set $\mathcal{W}_1$ is globally asymptotically stable.

For the local systems, $|f_2(x,e)| \le L_0|e| + H_0(x)$ where $L_0 = 0.1$ and $H_0(x) := |x_1^2 - x_1^3 + x_2 - 0.2x_1|$. Choose $V_0(x) := 10x_1^2 + 40x_2^2$. Then

$$\begin{aligned}
\frac{\partial V_0(x)}{\partial x} f_1(x,e) &= 20x_1^3 - 20x_1^4 + 20x_1x_2 + 20x_1u - 80x_2^2 + 80x_1^2x_2 \\
&\le -V_0(x) + 60x_1^4 + 20x_1^3 + 20x_1^2 - 10x_2^2 + 20x_1u.
\end{aligned}$$

We can design $\rho_1(u, y) := 60x_1^4 + 20x_1^3 + 20x_1^2 + 20x_1u$.

For $x \in \{x' | V_0(x') \le 10^{-3}\}$, $\frac{\partial V_0(x)}{\partial x} f_1(x,e) \le -0.3x_1^2 - 30x_2^2 + 5e^2$.

Let $U_0(x,e,\tau) := V_0(x) + \max\{b|x_1|^2, \gamma_0\phi_0(\tau)W^2(e)\}$ where ϕ_0 satisfies $\dot{\phi}_0 = -2L_0\phi_0 - \gamma_0\phi_0^2 - \gamma_0$ with $\gamma_0 = 2.3$. Choose $b = 0.04$, $\phi_0(0) = 2$ and $\phi_0(T_0) \ge 0.5$. To compare fairly, we also choose $T_0 = T_1$ and $\widetilde{\sigma}_0(y) := \frac{1.4}{16}|x_1|$. It holds that $U_0^\circ \le -U_0 + 2x_1^6 + 62x_1^4 + 20.08x_1^3 + 19x_1^2 + 2W^2(e)$. Then the function ρ_0 can be chosen as $\rho_0(e, y) := 2x_1^6 + 62x_1^4 + 20.08x_1^3 + 19x_1^2 + 2W^2(e)$.

When $U_0(x,e,\tau) \le 10^{-3}$, we can verify $U_0^\circ \le -\alpha_0(|(x,e,\tau)|_{\mathcal{W}_0})$ for some $\alpha_0 \in \mathcal{K}_\infty$ and $U_0(x^+, e^+, \tau^+) \le U_0(x,e,\tau)$ for $\tau = T_0$. Choose $\varepsilon_{1b} := 4 \times 10^{-4}$, $\varepsilon_{0b} := 3 \times 10^{-4}$, $\varepsilon_{1a} := 10^{-3}$ and $\varepsilon_{0a} := 9 \times 10^{-4}$ such that conditions (2) and (5) of Assumption 4.2 hold. The state trajectories are plotted in Figs. 4.7 and 4.9. The transmission instants and intervals are plotted in Figs. 4.8 and 4.10. The logic variable is plotted in Fig. 4.11. It can be seen that with the proposed hybrid algorithm,

Fig. 4.7 State trajectory with global controller

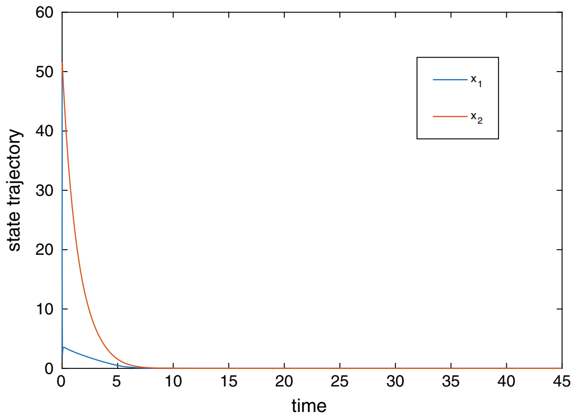


Fig. 4.8 Transmission instants and intervals with global controller

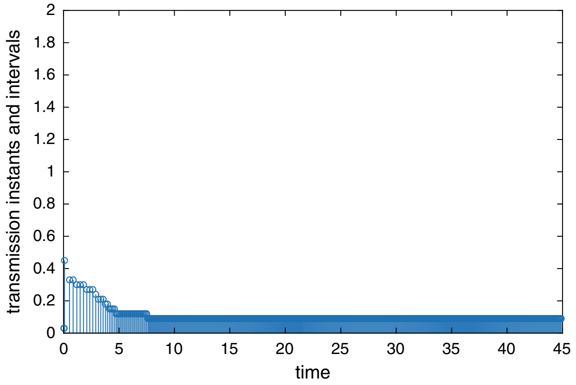
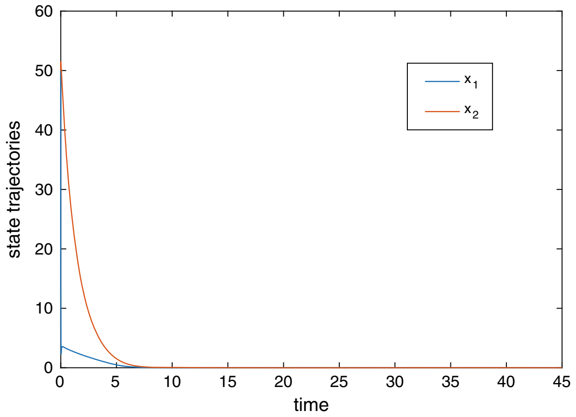


Fig. 4.9 State trajectories with hybrid controller



the global controller is eventually replaced by the local one and the state value will enter the attraction domain of the local system. The transmissions are significantly reduced with the hybrid transmission strategy.

Fig. 4.10 Transmission instants and intervals with hybrid controller

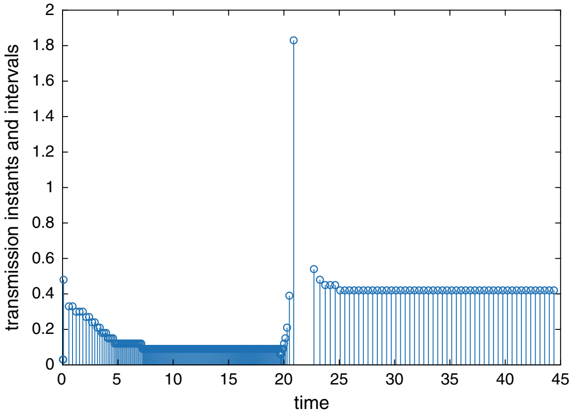
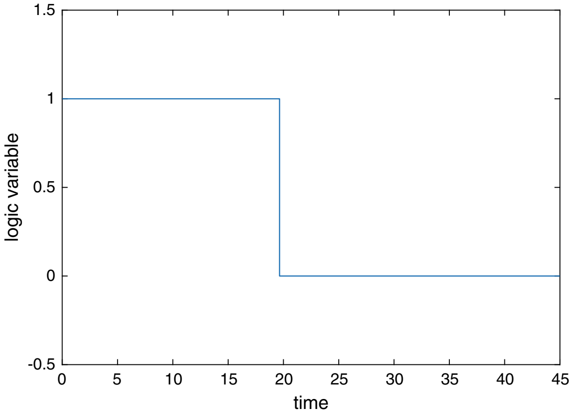


Fig. 4.11 Logic variable with hybrid controller



4.5 Conclusions

This chapter proposes a hybrid event-triggered controller for nonlinear networked control systems. The controller consists of a global periodic event-triggered controller and a local periodic event-triggered controller. A hybrid algorithm is proposed to determine when a switch between the local system and global system happens. The resulting system is modeled as a hybrid system which satisfies the hybrid basic conditions. Global asymptotic stability and robust stability are concluded.

Chapter 5
Event-Triggered Control for Nonlinear Systems With Stochastic Dynamics, Transmission Instants and Protocols

5.1 Introduction

The event-triggered transmission strategies in [21, 80] require continuous monitoring of the inequality conditions, and the Zeno phenomenon is precluded by showing that there is a lower bound between two consecutive transmissions. Differently, the periodic event-triggered transmission [27, 28, 86, 107] does not need to monitor the triggering condition continuously, but checks the triggering condition at discrete times. In [28, 107], periodic event-triggered transmission strategies are proposed for linear systems, where the detecting actions are taken periodically. In [86], the idea of a periodic event-triggered transmission strategy is extended to nonlinear systems and the closed-loop networked control system is modeled as a hybrid system. Stability conditions are derived by constructing a Lyapunov function where the timer variable and logic variable are considered part of the system state.

In the periodic event-triggered transmission [28, 86, 107], the triggering condition is checked periodically which means that the time intervals between two consecutive detecting instants are constant. Another possibility is to consider the situation where the discrete detecting instants are stochastically distributed. Several observations make stochastic detecting instants reasonable and meaningful. Firstly, in the real world, the timer that generates the periodic detecting signals has an accuracy error that may be stochastically distributed. Secondly, stochastically distributed detecting instants allow a relatively large duration of the detecting period with low probability. A similar idea has been considered in [30, 33], which do not consider the event-triggered control problem.

Motivated by the above observations, this chapter aims at analyzing stability of stochastic event-triggered nonlinear control systems with stochastic noise and protocols [9, 33, 93]. Stochastic event-triggered transmission implies that the discrete detecting instants are generated by a stochastic timer signal, which allows that the detecting interval is relatively large with low probability and relatively small with high probability. Besides, multiple network nodes are also considered and, at a transmission time, which node is transmitted is determined by a stochastic transmission

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protocol. The challenging point of this work is how to model and analyze the entire closed-loop system exhibiting stochastic behavior and hybrid behavior.

The main content of this chapter can be summarized as follows. Firstly, a stochastic event-triggered transmission strategy is proposed for nonlinear stochastic systems under stochastic transmission protocols in the network. A novel stochastic networked control system model is established based on the stochastic hybrid system framework proposed in [83]. Secondly, a novel twice continuously differentiable Lyapunov function is constructed, which can be used to deal with stochastic noise in the plant. The Lyapunov function contains a timer variable which is regarded as part of the system state. Stability conditions are derived to guarantee asymptotic stability in probability for the resulting system.

5.2 Model Description

Consider the following stochastic system

$$dx_p = f_p(x_p, u)dt + l_p(x_p, u)dw, \quad y = h_p(x_p) \tag{5.1}$$

where $x_p \in \mathbb{R}^{n_{x_p}}$ is the state, $y \in \mathbb{R}^{n_y}$ is the output, $u \in \mathbb{R}^{n_u}$ is the control input, and w denotes an n_w-dimensional Brownian motion. The output function h_p can be written as $h_p(x_p) := (\bar{h}_1(x_p), \ldots, \bar{h}_{n_y}(x_p)) \in \mathbb{R}^{n_y}$ and each $\bar{h}_i$ is assumed to be twice continuously differentiable.

The output of system (5.1) is monitored locally and when a triggering condition is satisfied, the output is transmitted via a communication network to a remote controller. The network-based controller is given as follows

$$\dot{x}_c = f_c(x_c, \hat{y}), \quad u = k(x_c, \hat{y}) \tag{5.2}$$

with $\hat{y} \in \mathbb{R}^{n_y}$. $\hat{y}$ will be updated at discrete transmission instants which are determined by the stochastic event-triggered transmission strategy and, between two consecutive transmission instants, $\hat{y}$ is kept constant by a zero-order-hold. Let $e := \hat{y} - y \in \mathbb{R}^{n_e}$ with $n_e = n_y$. Assume that there are ℓ nodes in the network and y can be partitioned into $y := (y_1, y_2, \ldots, y_\ell)$ with $y_k \in \mathbb{R}^{n_{y_k}}$ correspondingly. Then e can also be partitioned into $(e_1, e_2, \ldots, e_\ell)$ with $e_k \in \mathbb{R}^{n_{e_k}}$ and $n_{e_k} = n_{y_k}$.

A triggering condition will be checked at stochastically distributed detecting instants by a local event monitor. If the triggering condition is satisfied at some detecting time, some node j will have access to the network, and the data y_j collected by the sensors is transmitted to the controller. Which node has access to the network is determined by a stochastic transmission protocol. Specifically, if the triggering condition is satisfied at some detection instant, then $\hat{y}$ will be updated according to

$$\hat{y}^+ = y + h(e, v_1) \tag{5.3}$$

where v_1 is a random variable taking values in $\{1, 2, \ldots, \ell\}$ with distribution μ_1. $h(e, v_1)$ is defined as follows

$$h(e, v_1) := (I - \Gamma_{v_1})e$$

where I is the identity matrix and Γ_{v_1} is a block diagonal matrix

$$\Gamma_{v_1} := \mathrm{diag}\{0_{n_{e_1}}, \ldots, 0_{n_{e_{v_1-1}}}, I, 0_{n_{e_{v_1+1}}}, \ldots, 0_{n_{e_\ell}}\}$$

with $0_{n_{e_i}}$ being the $n_{e_i} \times n_{e_i}$ zero matrix. The Eq. (5.3) can be written as

$$e^+ = h(e, v_1). \tag{5.4}$$

The updating Eq. (5.4) is seen as the stochastic transmission protocol equation. Assume that the stochastic transmission protocols are stable in the following sense (see [79]).

Assumption 5.1 There exist a twice continuously differentiable function $W : \mathbb{R}^{n_e} \to \mathbb{R}$, constants $a_1, a_2 > 0$ and $\lambda \in (0, 1)$ such that the following inequalities hold for all $e \in \mathbb{R}^{n_e}$

$$\begin{aligned} &a_1|e|^2 \le W(e) \le a_2|e|^2 \\ &\sum_{k=1}^{\ell} W(h(e,k))\mu_1(k) \le \lambda W(e). \end{aligned} \tag{5.5}$$

Remark 5.1 Consider a quadratic function $W(e) := e^T Qe$ where Q is a positive definite diagonal matrix. Then we have

$$\sum_{k=1}^{\ell} W(h(e,k))\mu_1(k) \le \lambda W(e)$$

with $\lambda = (1 - \min_{k\in\{1,2,\ldots,\ell\}} \mu_1(k))$, which satisfies Assumption 5.1 if $\mu_1(k) \in (0, 1)$ for any $k \in \{1, 2, \ldots, \ell\}$. It is also interesting to consider the other stochastic protocols, for example, Markov scheduling protocols.

At stochastically distributed discrete detecting instants, the following triggering condition will be checked

$$W(e) \ge q\sigma(y) \tag{5.6}$$

where $\sigma : \mathbb{R}^{n_y} \to \mathbb{R}_{\ge 0}$ is a continuous function, and $q \ge 0$ is a coefficient.

Introduce a timer variable $\tau \in \mathbb{R}$ that will be used to generate stochastically distributed discrete detecting instants, and denote $\chi := (x, e, \tau)$. The entire closed-loop system can be modeled into the following stochastic hybrid system

$$\begin{cases} d\chi = F(\chi)dt + B(\chi)dw, & \chi \in C \\ \chi^+ \in G(\chi, v^+), & \chi \in D. \end{cases} \tag{5.7}$$

The flow set C is specified as $C := \{\chi | \tau \in [0, T_{max}]\}$ and the jump set D is specified as $D := \{\chi | \tau = 0\}$.

The flow dynamics are specified as follows

$$\begin{cases} dx = f_1(x, e)dt + b_1(x, e)dw \\ de = f_2(x, e)dt + b_2(x, e)dw \\ d\tau = -dt \end{cases} \tag{5.8}$$

where $f_2(x, e) := (f_{21}(x, e), \dots, f_{2n_y}(x, e)) \in \mathbb{R}^{n_y}$,

$$\begin{aligned} f_1(x, e) &:= \begin{pmatrix} f_p(x_p, k(x_c, h_p(x_p) + e)) \\ f_c(x_c, e + h_p(x_p)) \end{pmatrix} \\ f_{2i}(x, e) &:= -\frac{\partial \bar{h}_i(x_p)}{\partial x_p} f_p(x_p, k(x_c, h_p(x_p) + e)) - \frac{1}{2}\mathrm{Tr}(l_p^T(\nabla^2 \bar{h}_i)l_p) \\ b_1(x, e) &:= \begin{pmatrix} l_p(x_p, k(x_c, h_p(x_p) + e)) \\ 0 \end{pmatrix} \\ b_2(x, e) &:= -\frac{\partial h_p(x_p)}{\partial x_p} l_p(x_p, k(x_c, h_p(x_p) + e)). \end{aligned}$$

and l_p in f_{2i} is $l_p(x_p, k(x_c, e + h_p(x_p)))$. The jump map is specified as

$$G(\chi, v) := \begin{cases} (x, e, v_2), & W(e) < q\sigma(y) \\ (x, h(e, v_1), v_2), & W(e) > q\sigma(y) \\ (x, \{e, h(e, v_1)\}, v_2), & W(e) = q\sigma(y) \end{cases} \tag{5.9}$$

where v_1 and v_2 are two independent random variables. From the model, once the stochastic timer reaches zero, the triggering condition (5.6) will be checked and the jump map captures all the possible cases.

μ_1 and μ_2 are respectively the distribution functions of v_1 and v_2. The distributed functions μ_1 and μ_2 satisfy that

$$\mu_1(\{1, \dots, \ell\}) = 1, \quad \mu_2([T_{min}, T_{max}]) = 1 \tag{5.10}$$

with the parameters satisfying $0 < T_{min} \leq T_{max}$.

To clarify how the model generates the stochastically distributed detecting times note that, at each reset time, the timer τ is set to a random value in $[T_{min}, T_{max}]$ almost surely and then, due to the timer's continuous-time dynamics and the definition of the jump set, the next reset time is equal to the current reset time plus this random value.

It should be noted that the inequality (5.6) can be checked by the event monitor because the event monitor can access to the output vector without network communication if the nodes are located near each other. This setup has been adopted in [14, 86].

5.3 Stability Analysis

With the function W in Assumption 5.1, two coupling conditions are imposed for stability analysis.

Condition 5.1 There exist a constant $L > 0$ and two functions $H : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $N : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the following inequality holds for all $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$

$$\frac{\partial W(e)}{\partial e} f_2(x, e) + \frac{1}{2}\mathrm{Tr}(b_2^T(\nabla^2 W)b_2) \leq LW(e) + 2\sqrt{W(e)}H(x) + N(x). \quad (5.11)$$

Remark 5.2 If $b_2 = 0$, then $N(x) = 0$. Moreover, if W is a quadratic function, then Condition 5.1 can be easily satisfied if there exist a number $M > 0$ and a function $Y : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ such that $|f_2(x, e)| \leq M|e| + Y(x)$. Detailed clarifications for Condition 5.1 have been given in the example section.

Condition 5.2 There exist a positive definite function $\tilde{\sigma} : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, a function $\delta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a constant $K \geq 0$ such that for all $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$, the following inequalities hold:

$$\tilde{\sigma}(x) \geq \sigma(C_p x_p)$$

$$\frac{\partial \tilde{\sigma}(x)}{\partial x} f_1(x, e) + \frac{1}{2}\mathrm{Tr}(b_1^T(\nabla^2 \tilde{\sigma})b_1) \leq \delta(x) + KW(e).$$

Next, we impose a stability assumption for the x subsystem, which will be used to construct a Lyapunov function for the entire closed-loop system by combining Assumption 5.1 and the coupling conditions Conditions 5.1 and 5.2. Assumption 5.2 involves coupling with some of the functions that appear in the two previous coupling conditions.

Assumption 5.2 Suppose that there exist a twice continuously differentiable function $V : \mathbb{R}^{n_x} \to \mathbb{R}$, $\alpha_i \in \mathcal{K}_\infty (i = 1, 2)$, a positive definite function $\varepsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and constants $\theta > 0$, $\gamma > 0$ such that the following conditions hold:

- $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ for all $x \in \mathbb{R}^{n_x}$,

- for all $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$, the following inequality holds

$$\begin{aligned}\frac{\partial V(x)}{\partial x} f_1(x, e) + \frac{1}{2}\mathrm{Tr}(b_1^T(\nabla^2 V)b_1) \leq {} & -\varepsilon(|(x, e)|) - H^2(x) - \theta N(x) \\ & - \tilde{\sigma}(x) - \delta(x) + \gamma^2 W(e).\end{aligned}$$

Remark 5.3 ε satisfies $\varepsilon(0) = 0$ and $\varepsilon(s) > 0$ for $s > 0$. Assumption 5.2 can be satisfied for linear systems if x subsystem is stable. How to compute the parameters will be explained in the example section.

Consider the following scalar differential equation

$$\dot{\phi} = L\phi + \gamma\phi^2 + \gamma + \frac{K}{\gamma} \tag{5.12}$$

with the initial value satisfying $\phi(0) = \rho \in (0, \frac{\theta}{\gamma})$, and ϕ is defined on $[0, \mathscr{T})$ where

$$\mathscr{T} := \begin{cases} \frac{1}{r_2\sqrt{\gamma}}\left(\frac{\pi}{2} - \arctan\frac{\sqrt{\gamma}(\rho + r_1)}{r_2}\right), & \gamma + \frac{K}{\gamma} > \frac{L^2}{4\gamma} \\ \frac{1}{\gamma}\frac{1}{\rho + r_1}, & \gamma + \frac{K}{\gamma} = \frac{L^2}{4\gamma} \\ \frac{1}{2r_2\sqrt{\gamma}} \ln\left|\frac{\sqrt{\gamma}(\rho + r_1) + r_2}{\sqrt{\gamma}(\rho + r_1) - r_2}\right|, & \gamma + \frac{K}{\gamma} < \frac{L^2}{4\gamma} \end{cases} \tag{5.13}$$

with $r_1 := \frac{L}{2\gamma}$ and $r_2 := \sqrt{|\gamma + \frac{K}{\gamma} - \frac{L^2}{4\gamma}|}$.

The number $\mathscr{T}$ in (5.13) is such that $\lim_{s \uparrow \mathscr{T}} \phi(s) = \infty$. For a twice continuously differentiable function U, define the following operator

$$\mathscr{L}U(\chi) := \frac{\partial U(\chi)}{\partial \chi} F(\chi) + \frac{1}{2}\mathrm{Tr}(B^T(\chi)(\nabla^2 U)B(\chi)).$$

We are ready to give the main result of this chapter.

Theorem 5.1 *Consider the system (5.7). Assume Assumptions 5.1 and 5.2 and Conditions 5.1 and 5.2 hold. Suppose that* $0 < T_{min} \leq T_{max} < \mathscr{T}$ *and the following conditions hold:*

(1) the distribution functions of the random variable v_1 *and* v_2 *satisfy (5.10) and the following condition*

$$\int_{[T_{min}, T_{max}]} \phi(\mu_2)\mu_2(dv_2) < \frac{\rho}{\lambda},$$

(2) $\gamma\phi(T_{max}) \le \theta$,
(3)

$$0 \le q < \frac{1}{\gamma}\frac{1 - \int_{[T_{min},T_{max}]} e^{-v_2}\mu_2(dv_2)}{\int_{[T_{min},T_{max}]} \phi(v_2)\mu_2(dv_2) - \rho}.$$

Then the set $\mathscr{W} := \{0\} \times \{0\} \times [0, T_{\max}]$ *is UGASp.*

Remark 5.4 Because $T_{\min} > 0$ and the probability outside the set $[T_{\min}, T_{\max}]$ is zero for the random variable v_2 according to Condition (1) of Theorem 5.1, Zeno solutions do not happen almost surely.

Remark 5.5 If $N(x) = 0$ for all $x \in \mathbb{R}^{n_x}$ which often holds for the case without stochastic noise in the plant, then condition (2) of Theorem 5.1 can be removed since θ can be taken arbitrarily large.

Remark 5.6 Because $\phi(0) = \rho < \frac{\rho}{\lambda}$ and $\rho < \frac{\theta}{\gamma}$, conditions (1) and (2) of Theorem 5.1 can be satisfied for T_{max} sufficiently small. Besides, T_{max} should satisfy $T_{max} < \mathscr{T}$ to make ϕ well defined. Therefore, Theorem 5.1 allows that the stochastic networked control system is stable in probability provided that the detection interval is relatively large with low probability and relatively small with high probability. We note that in [30], the authors consider similar idea while in the deterministic manner for linear systems. The sampling interval can be very large in [30] but the average sampling interval is small.

Proof Construct a function as follows

$$U(\chi) := V(x) + e^{-\tau}\tilde{\sigma}(x) + \gamma\phi(\tau)W(e) \tag{5.14}$$

where ϕ is the solution of system (5.12).

Then we can compute for all $\chi \in C$ that

$$\begin{aligned}\mathscr{L}U(\chi) = &\frac{\partial V}{\partial x} f_1(x, e) + \frac{1}{2}\mathrm{Tr}(b_1^T(\nabla^2 V)b_1)\\ &+ e^{-\tau}\tilde{\sigma}(x) + e^{-\tau}\frac{\partial\tilde{\sigma}(x)}{\partial x} f_1(x, e)\\ &+ e^{-\tau}\frac{1}{2}\mathrm{Tr}(b_1^T(\nabla^2\tilde{\sigma})b_1) + \gamma\dot{\phi}(\tau)W(e)\\ &+ \gamma\phi(\tau)\left(\frac{\partial W(e)}{\partial e} f_2(x, e) + \frac{1}{2}\mathrm{Tr}(b_2^T(\nabla^2 W)b_2)\right).\end{aligned} \tag{5.15}$$

Combining Assumption 5.2 and Conditions 5.1 and 5.2, we can derive from (5.15) that

$$\begin{aligned}\mathscr{L}U(\chi) &\leq -\varepsilon(|(x,e)|) - H^2(x) - \theta N(x) - \tilde{\sigma}(x) - \delta(x)\\ &\quad + \gamma^2 W(e) + e^{-\tau}\tilde{\sigma}(x) + \delta(x) + KW(e)\\ &\quad + \gamma(-L\phi - \gamma\phi^2 - \gamma - \frac{K}{\gamma})W(e)\\ &\quad + \gamma\phi(LW(e) + 2\sqrt{W(e)}H(x) + N(x))\\ &\leq -\varepsilon(|(x,e)|) - H^2(x) - \gamma^2\phi^2 W(e) + 2\gamma\phi\sqrt{W(e)}H(x)\\ &\leq -\varepsilon(|(x,e)|).\end{aligned}$$

For $\chi \in D$, we next discuss three cases.

Case (I) $W(e) < q\sigma(y)$

We can derive for $g = G(\chi, v)$ that

$$\begin{aligned}\int_{\mathbb{R}^2} U(g)\mu(dv) - U(\chi) &= \tilde{\sigma}(x)\int_{[T_{min},T_{max}]} e^{-v_2}\mu_2(dv_2)\\ &\quad + \gamma\int_{[T_{min},T_{max}]}\phi(v_2)\mu_2(dv_2)W(e) - \tilde{\sigma}(x) - \gamma\rho W(e)\\ &\leq \tilde{\sigma}(x)\left(\int_{[T_{min},T_{max}]} e^{-v_2}\mu_2(dv_2) - 1\right)\\ &\quad + \gamma q\tilde{\sigma}(x)\left(\int_{[T_{min},T_{max}]}\phi(v_2)\mu_2(dv_2) - \rho\right)\\ &\leq -\varepsilon_1(|(x,e)|) \qquad (5.16)\end{aligned}$$

for some class-$\mathscr{K}$ function ε_1, where we have used condition 3) of Theorem 5.1, $\mu_2([T_{min}, T_{max}]) = 1$ and $\phi(s) > \rho$ for $s \in [T_{min}, T_{max}]$.

Case (II) $W(e) > q\sigma(y)$

For this case, we can derive

$$\begin{aligned}\int_{\mathbb{R}^2} U(g)\mu(dv) - U(\chi) &= \tilde{\sigma}(x)\int_{[T_{min},T_{max}]} e^{-v_2}\mu_2(dv_2) - \tilde{\sigma}(x) - \gamma\rho W(e)\\ &\quad + \gamma\int_{[T_{min},T_{max}]}\phi(v_2)\mu_2(dv_2)\sum_{k=0}^{\ell} W(h(e,k))\mu_1(k)\\ &\leq -\tilde{\sigma}(x)\left(1 - \int_{[T_{min},T_{max}]} e^{-v_2}\mu_2(dv_2)\right)\\ &\quad - \gamma W(e)\left(\rho - \lambda\int_{[T_{min},T_{max}]}\phi(v_2)\mu_2(dv_2)\right)\\ &\leq -\varepsilon_2(|(x,e)|) \qquad (5.17)\end{aligned}$$

for some class-$\mathscr{K}$ function ε_2, where we have used condition (1) of Theorem 5.1, Assumption 5.1, $\mu_2([T_{min}, T_{max}]) = 1$ and the independence of v_1 and v_2.

Case (III) $W(e) = q\sigma(y)$

Combining case (I) and case (II), we can also conclude some class-$\mathscr{K}$ function ε_3 such that

$$\int_{\mathbb{R}^2} \max_{g \in G(\chi, v)} U(g)\mu(dv) - U(\chi) \leq -\varepsilon_3(|(x, e)|). \tag{5.18}$$

By applying Corollary 1 of [83], the proof is completed. □

We next present a corollary for the special case of periodic sampling, which means $P(v_2 = T) = 1$.

Corollary 5.1 *Consider the system (5.7). Assume Assumptions 5.1 and 5.2 and Conditions 5.1 and 5.2 hold. Suppose that the stochastic detection reduces to the periodic detection with detection period* $T > 0$. *Then the set* $\mathscr{W} := \{0\} \times \{0\} \times [0, T_{\max}]$ *is UGASp if*

(1) the detection period satisfies $\phi(T) < \frac{\rho}{\lambda}$,

(2) $\gamma\phi(T) \leq \theta$,

(3) q *satisfies the following inequality*

$$0 \leq q < \frac{1}{\gamma} \frac{1 - e^{-T}}{\phi(T) - \rho}.$$

5.4 Example Studies

In this section, we show how to verify the conditions of Theorem 5.1 for linear systems and a nonlinear system.

5.4.1 Linear Systems

Consider the following linear plant

$$dx_p = A_p x_p dt + B_p u dt + E_p x_p dw, \qquad y = C_p x_p.$$

An observer-based controller is given as follows

$$dx_c = A_p x_c dt + B_p u dt + L_c(C_p x_c - \hat{y})dt, \ \ u = K_c x_c.$$

Then the closed-loop networked control system can be written in the form of the stochastic hybrid system (5.7) with

$$f_1(x,e) = A_{11}x + A_{12}e, \quad f_2(x,e) = A_{21}x$$
$$b_1(x,e) = A_{13}x, \quad b_2(x,e) = A_{23}x$$

and

$$\begin{aligned}
A_{11} &= \begin{pmatrix} A_p & B_pK_c \\ -L_cC_p & A_p + B_pK_c + L_cC_p \end{pmatrix}, \; A_{12} = \begin{pmatrix} 0 \\ -L_c \end{pmatrix} \\
A_{21} &= -\begin{pmatrix} C_pA_p & C_pB_pK_c \end{pmatrix}, \quad A_{13} = \begin{pmatrix} E_p & 0 \\ 0 & 0 \end{pmatrix} \\
A_{23} &= -C_p \begin{pmatrix} E_p & 0 \end{pmatrix}.
\end{aligned} \tag{5.19}$$

Choose a function $W(e) := e^T Qe$ where Q is a positive definite diagonal matrix. Assumption 5.1 can be satisfied according to Remark 5.1.

It is obvious that W is twice continuously differentiable. We can compute that

$$\begin{aligned}
\frac{\partial W(e)}{\partial e} f_2(x,e) + \frac{1}{2}\mathrm{Tr}(b_2^T(\nabla^2 W)b_2) &= 2e^T QA_{21}x + x^T A_{23}^T QA_{23}x \\
&\leq 2\sqrt{W(e)}|Q^{\frac{1}{2}}A_{21}x| + x^T A_{23}^T QA_{23}x.
\end{aligned}$$

Then for the linear case, the inequality (5.11) in Condition 5.1 can be satisfied with arbitrarily small $L > 0$ and

$$H(x) = |Q^{\frac{1}{2}}A_{21}x|, \quad N(x) = x^T A_{23}^T QA_{23}x. \tag{5.20}$$

Choose $\sigma(y) := l|C_px_p|^2$. Denote $S := \begin{pmatrix} C_p & 0 \end{pmatrix}$. Then $\sigma(y) = lx^TS^TSx$. Let $\tilde{\sigma}(x) := lx^TS^TSx + \mu|x|^2$. Then $\tilde{\sigma}$ is a positive definite function with $\mu > 0$. We compute that

$$\begin{aligned}
&\frac{\partial \tilde{\sigma}(x)}{\partial x} f_1(x,e) + \frac{1}{2}\mathrm{Tr}(b_1^T(\nabla^2\tilde{\sigma})b_1) \\
&= 2lx^TS^TS(A_{11}x + A_{12}e) + 2\mu x^T(A_{11}x + A_{12}e) \\
&\quad + lx^TA_{13}^TS^TSA_{13}x + \mu x^TA_{13}^TA_{13}x.
\end{aligned} \tag{5.21}$$

Let δ in Condition 5.2 be a quadratic function with $\delta(x) := d|x|^2$, then we can find $K \geq 0$ and $d \geq 0$ such that Condition 5.2 holds by solving the following linear matrix inequality

$$\begin{pmatrix} -d + \Phi_1 & \Phi_2 \\ \Phi_2^T & -K \end{pmatrix} \leq 0 \tag{5.22}$$

with

$$\Phi_1 = lS^T SA_{11} + lA_{11}^T S^T S + \mu A_{11} + \mu A_{11}^T + lA_{13}^T S^T SA_{13} + \mu A_{13}^T A_{13}$$
$$\Phi_2 = lS^T SA_{12} Q^{-\frac{1}{2}} + \mu A_{12} Q^{-\frac{1}{2}}.$$

Choose $V(x) := x^T Px$. Then we can compute that

$$\frac{\partial V(x)}{\partial x} f_1(x, e) + \frac{1}{2}\mathrm{Tr}(b_1^T (\nabla^2 V)b_1) = 2x^T P(A_{11}x + A_{12}e) + x^T A_{13}^T PA_{13}x. \tag{5.23}$$

Define $\varepsilon(|(x, e)|) := \epsilon^*|x|^2 + \epsilon^*|e|^2$. With $N(x)$, $H(x)$, $\delta(x)$ and $\tilde{\sigma}(x)$ at hand, Assumption 5.2 can be transformed into solving the following linear matrix inequality with a matrix variable $P > 0$ and a scalar variable $\gamma > 0$ by choosing the parameters $\epsilon^* > 0$, $\mu > 0$, $l > 0$ and $\theta > 0$ in prior

$$\begin{pmatrix} \Psi & PA_{12}Q^{-\frac{1}{2}} \\ Q^{-\frac{1}{2}}A_{12}^T P & -\gamma^2 I + \epsilon^* Q^{-1} \end{pmatrix} \leq 0$$

where

$$\Psi = PA_{11} + A_{11}^T P + A_{13}^T PA_{13} + (\epsilon^* + d + \mu)I + A_{21}^T QA_{21} + \theta A_{23}^T QA_{23} + lS^T S.$$

With the above observations at hand, Assumption 5.2 can be satisfied for linear systems provided that $PA_{11} + A_{11}^T P + A_{13}^T PA_{13} < 0$ and the parameters can be chosen by solving linear matrix inequalities.

5.4.2 Nonlinear Systems

Consider the following system

$$\begin{cases} dx_1 = (x_1^2 - x_1^3 + x_2 + u_1)dt + 0.1x_1 dw \\ dx_2 = (x_2^2 - x_2^3 + x_1 + u_2)dt + 0.1x_2 dw \end{cases}$$

with output $y = (x_1, x_2)$. Assume that there are two nodes in the network with $y_1 = x_1$ and $y_2 = x_2$, and the probability distribution functions for the two node are $\mu_1(1) = 0.45$ and $\mu_1(2) = 0.55$. Consider the controller $u_1 = -2\hat{y}_1,\ u_2 = -2\hat{y}_2$. Then the closed-loop system can also be written into the form of system (5.7). The flow dynamics of (x, e) can be written as

$$\begin{cases} dx_1 = (-2x_1 + x_1^2 - x_1^3 + x_2 - 2e_1)dt + 0.1x_1 dw \\ dx_2 = (-2x_2 + x_2^2 - x_2^3 + x_1 - 2e_2)dt + 0.1x_2 dw \\ de_1 = (2x_1 - x_1^2 + x_1^3 - x_2 + 2e_1)dt - 0.1x_1 dw \\ de_2 = (2x_2 - x_2^2 + x_2^3 - x_1 + 2e_2)dt - 0.1x_2 dw. \end{cases}$$

Choose $W(e) := e^T e$. It is easy to get $\lambda = 0.55$, $L = 4$, $N(x) = 0.02x_1^2 + 0.02x_2^2$ and

$$H(x) = \sqrt{(2x_1 - x_1^2 + x_1^3 - x_2)^2 + (2x_2 - x_2^2 + x_2^3 - x_1)^2}.$$

Choose the candidate Lyapunov function for x subsystem as $V(x) := 2x_1^2 + 2x_2^2 + \frac{1}{2}x_1^4 + \frac{1}{2}x_2^4$. It holds by calculation that

$$\begin{aligned} \frac{\partial V(x)}{\partial x} f_1(x, e) + \frac{1}{2}\mathrm{Tr}(b_1^T(\nabla^2 V)b_1) \leq & -1.5x_1^2 - 1.5x_2^2 - 0.15x_1^4 - 0.15x_2^4 \\ & + 36e_1^2 + 36e_2^2 - H^2(x). \end{aligned}$$

Set $\tilde{\sigma}(x) = \sigma(x) = 0.2|x|^2$. We obtain $\delta(x) = 0.3|x|^2$ and $K = 0.4$. We thus take $\theta = 30$ and $\rho = \sqrt{0.55}$. Assume $T_{min} = 0.001$, $T_{max} = 1$ and the detection period is subject to the truncated normal distribution over $[T_{min}, T_{max}]$. Then the probability dense function of v_2 satisfies

$$p_2(v_2) = \frac{\frac{1}{\sqrt{2\pi}b} e^{-\frac{(v_2 - a)^2}{2b^2}}}{\varphi(T_{max}) - \varphi(T_{min})}, \quad v_2 \in [T_{min}, T_{max}]$$

and $p_2(v_2) = 0$ for $v_2 \in R - [T_{min}, T_{max}]$ where φ is the cumulative distributed function of a normal distribution with mean a and variance b, and is given as follows

$$\varphi(t) := \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}b} e^{-\frac{(s-a)^2}{2b^2}} ds.$$

Set $a = 0.01$ and $b = 0.025$. We compute $q < 0.0093$ by Theorem 5.1. We run a simulation for 25 s. A typical sample path of the state is plotted in Fig. 5.1. The detecting instants and detecting intervals generated by the timer are plotted in Fig. 5.2. There are 1009 detecting instants. The transmission instants and transmission intervals are plotted in Fig. 5.3. There are 807 transmissions among 25 s and the average transmission interval is 0.031 s. There are 371 transmissions for the first node and 436 transmissions for the second node. The simulation result illustrates the effectiveness of the proposed theorem. Besides, it can be seen from Fig. 5.2 that the largest detecting interval is about 0.1 s while we compute by Corollary 5.1 that the detecting period is 0.0356 s for periodic detecting. This shows that the stochastic detecting interval has a high likelihood of short time and low likelihood of long time.

Fig. 5.1 State trajectories

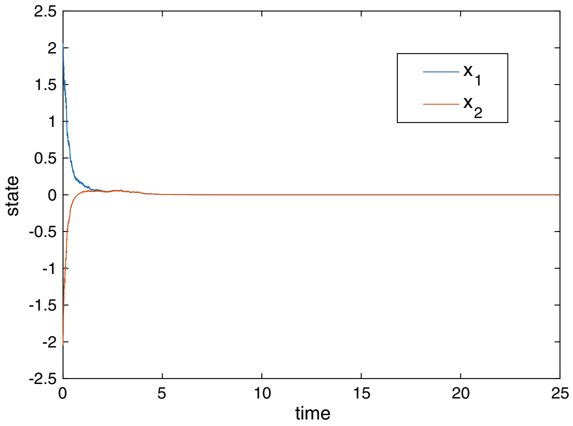


Fig. 5.2 Detecting instants and detecting intervals generated by the timer

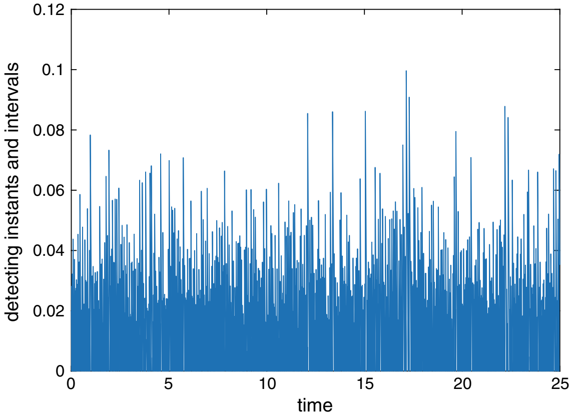
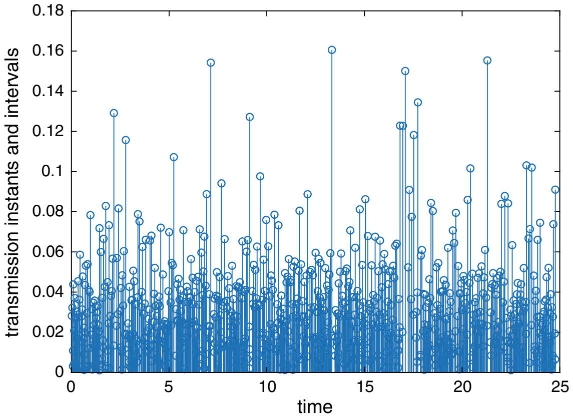


Fig. 5.3 Transmission instants and transmission intervals



5.5 Conclusions

The stability problem of the event-triggered control is considered for nonlinear systems with stochastic dynamics, transmission times and protocols in this chapter. The triggering condition is checked at stochastically distributed discrete instants. To analyze stability, the closed-loop system is modeled as a stochastic hybrid system. Stability conditions are derived by constructing a Lyapunov function.

Part II

Distributed Optimization with Network Communication

Chapter 6
Stability Analysis of Distributed Convex Optimization Under Persistent Attacks

6.1 Introduction

In this chapter, we explore the influence of potential attacks on the ability of an algorithm to identify the optimal solution of distributed unconstrained optimization problems and the considered attacks change only the communication topology. To study this problem, we model the distributed algorithm proposed in [37] as a switched algorithm. This algorithm is stable in the absence of attacks and it may be unstable in the presence of attacks. When attacks occur, we allow attackers to be able to switch topologies at will, until the attacks are shut down and normal operation resumes. That is, the topology has arbitrary time-varying edges when attackers are affecting the network. Some work has been done on the problem of distributed optimization under time-varying graphs by using discrete-time algorithms (see, for example, [61–63]). In [62], a discrete-time broadcast-based algorithm is developed to steer every node to an optimal value under an assumption of uniformly strongly connected graph sequence. In this chapter, we consider a continuous-time algorithm and establish an exponential convergence rate by combining hybrid system theory and Lyapunov function approaches. A differential inclusion is used to model attack modes (unstable modes). Differential inclusions can model arbitrary switching without having to specify the switching signal a priori. To guarantee exponential convergence to the optimal solution, we use an average dwell-time automaton and time-ratio monitor to constrain attacks. Some work related to average dwell-time automata and time-ratio constraints has been reported. For example, the paper [32] introduced an average dwell-time constraint to analyze stability for switched systems with stable subsystems. The paper [55] introduced time-ratio constraints to obtain input/output-to-state stability for switched nonlinear systems with stable and unstable subsystems. Work [8] used an average dwell-time automaton to cast switched systems with this type of constraint as a hybrid system. Following these contributions, some work has been done in other similar contexts (see, for example, [67, 89, 97]). The average dwell-time automaton is used to limit the frequency of attacks. The time-ratio constraint is used to limit the relative duration of the attacks. The frequency and relative duration

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of the attacks impact the ability of the algorithm to identify the optimal solution. In this chapter, we explore how large the frequency and relative duration can be while still guaranteeing convergence to the optimal solution.

6.2 Knowledge About Graph

Consider a weighted graph $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ consisting of a finite vertex set $\mathscr{V} := \{1, 2, \ldots, N\}$ and an edge set $\mathscr{E}$. $\mathscr{N}_i := \{j : (j, i) \in \mathscr{E}\}$ denotes the neighbors of vertex i. If agent i can obtain information from agent j, then $(j, i) \in \mathscr{E}$ and agent $j \in \mathscr{N}_i$. A path of graph $\mathscr{G}$ is a sequence of distinct agents in $\mathscr{V}$ such that any consecutive agents in the sequence correspond to an edge of graph $\mathscr{G}$. If there is a path between any two vertices of a graph $\mathscr{G}$, then the graph is connected.

$\mathscr{A} := [a_{ij}] \in \mathbb{R}^{N\times N}$ is a weighted adjacency matrix of $\mathscr{G}$ with $a_{ij} := 1$ if $j \in \mathscr{N}_i$ and $a_{ij} := 0$, otherwise. Define the degree matrix $D := diag\{\bar{d}_1, \ldots, \bar{d}_N\} \in \mathbb{R}^{N\times N}$, where $\bar{d}_i := \sum_{j=1}^{N} a_{ij}$ for $i = 1, \ldots, N$, and $L := D - \mathscr{A}$ is the Laplacian matrix of $\mathscr{G}$. For a connected undirected graph, the Laplacian matrix has a simple zero eigenvalue with the corresponding eigenvector space $\{\alpha \mathbf{1}_N | \alpha \in \mathbb{R}\}$, and $L\mathbf{1}_N = (\mathbf{1}_N^T L)^T = \mathbf{0}_N$, while all other eigenvalues are positive, that is $0 < \lambda_2 \leq \cdots \leq \lambda_N$.

Next some knowledge related to convex analysis is introduced.

A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be convex if

$$f(\alpha x+(1-\alpha)y) \leq \alpha f(x)+(1-\alpha)f(y), \forall x, y\in\mathbb{R}^m, \forall\alpha\in[0, 1].$$

The gradient function $\nabla f : \mathbb{R}^m \to \mathbb{R}^m$ is said to be Lipschitz continuous with constant $\ell > 0$ if

$$|\nabla f(x) - \nabla f(y)| \leq \ell|x - y|, \ \forall x, y \in \mathbb{R}^m. \tag{6.1}$$

A differentiable function f_i, $i \in N$ is strongly convex with constant $\bar{\kappa}_i > 0$ on $\mathbb{R}^m$ if

$$(\nabla f_i(x) - \nabla f_i(y))^T(x - y) \geq \bar{\kappa}_i|x - y|^2, \forall x, y \in \mathbb{R}^m. \tag{6.2}$$

6.3 Problem Formulation

Consider a network of N agents with interaction topology described by an undirected graph $\mathscr{G}$. Each agent i is equipped with a local objective function $f_i : \mathbb{R}^m \to \mathbb{R}$, which is privately observed and known only by agent i. The following assumption is needed throughout this chapter.

Assumption 6.1 The local objective functions $f_i : \mathbb{R}^m \to \mathbb{R}$, $i \in N$ are continuously differentiable, strongly convex with constants $\bar{\kappa}_i > 0$, and the gradient ∇f_i of each f_i is Lipschitz continuous with constant $\ell_i > 0$.

Our objective is to find a distributed algorithm such that agents solve

$$\min_{x \in \mathbb{R}^m} g(x) := \sum_{i=1}^{N} f_i(x) \tag{6.3}$$

using only their own local data and exchanged information with their neighbors. In order to solve this problem, like in [20], we create an equivalent optimization problem that involves N copies of the variable x and a constraint, expressed in terms of the Laplacian matrix, that forces all of these copies to be the same. In particular, letting $x_i \in \mathbb{R}^m$, denote the ith copy of x, $X := col(x_1, \ldots, x_N) \in \mathbb{R}^{Nm}$, and $\mathbf{L}_0 := L \otimes I_m \in \mathbb{R}^{Nm \times Nm}$, where $L \in \mathbb{R}^{N \times N}$ is the Laplacian matrix of $\mathscr{G}$, we consider the following optimization problem:

$$\begin{aligned} \min_{x_i \in \mathbb{R}^m} \tilde{f}(X) &:= \sum_{i=1}^{N} f_i(x_i), \\ \text{subject to} \quad \mathbf{L}_0 X &= \mathbf{0}_{Nm}. \end{aligned} \tag{6.4}$$

According to [20, Lemma 3.1], we have that the problem (6.3) on $\mathbb{R}^m$ is equivalent to the problem (6.4) on $\mathbb{R}^{Nm}$. Also, from [20] we know that the constrained optimization problem (6.4) is feasible under Assumption 6.1, i.e., the minimum is achieved for some X. The local objective function f_i is the private data for agent i, which is not shared with other agents. This makes (6.4) a distributed optimization problem. Therefore, the task for all agents is to cooperatively achieve consensus with the minimum global cost $\tilde{f}$ to solve problem (6.3).

In the ideal case, communication channels can transmit their information successfully and the optimal solution can be found easily (see, for example, [37, 42, 78, 108] and references cited therein). In practice, however, making all communication networks connected or secure is impossible since the existence of networked attacks may lead to communication failures (see [91]). If communication topologies break down, the behavior of an algorithm may be influenced. Therefore, it is necessary to analyze the influence on the ability of an algorithm to identify the optimal solution under persistent attacks. Here we use the set $\mathscr{Q} \subset \mathbb{Z}_{>0}$ to denote the types of possible persistent attacks and $\{0\}$ to denote that the topology does not encounter any attack.

Different types of attacks may destroy different edges of the communication network and cause instability of the optimization algorithms. To analyze the effect of attacks on the ability of an algorithm to identify the optimal solution, we model the algorithm proposed in [37] as a switched algorithm:

$$\begin{aligned}\dot{x}_i &= -\nabla f_i(x_i) - \sum_{j\in\mathscr{N}_i}(a_{ij})_\sigma(x_i - x_j) - z_i \\ \dot{z}_i &= \sum_{j\in\mathscr{N}_i}(a_{ij})_\sigma(x_i - x_j),\ Z \in \mathbb{A},\end{aligned} \tag{6.5}$$

where $z_i \in \mathbb{R}^m$ is the auxiliary variable with initial value $z_i(0)$, $Z := col(z_1, \dots, z_N)$, $Z \in \mathbb{A} := \{Z \in \mathbb{R}^{Nm} | \sum_{i=1}^N z_i = \mathbf{0}_m\}$. It can be shown that $\mathbb{A}$ is forward invariant. $(a_{ij})_\sigma \in \{0, 1\}$, and $\sigma : [t_0, +\infty) \to \mathscr{P}$ is a piecewise constant function with $\mathscr{P} := \{0\} \cup \mathscr{Q}$. For each agent $i \in \mathscr{V}$, the algorithm is starting from $x_i(0), z_i(0) \in \mathbb{R}^m$ with $\sum_{i=1}^N z_i(0) = \mathbf{0}_m$. Denote $\nabla\tilde{f}(X) := col(\nabla f_1(x_1), \dots, \nabla f_N(x_N)) \in \mathbb{R}^{Nm}$. $\mathbf{L}_\sigma := L_\sigma \otimes I_m \in \mathbb{R}^{Nm\times Nm}$, where L_σ denote the Laplacian matrices of topologies. When there does not exist any kind of attack (or $\sigma = 0$), we can rewrite the algorithm (6.5) in a Kronecker product form:

$$\dot{\eta} = \bar{F}_0(\eta), \tag{6.6}$$

where $\eta := col(X, Z)$, $\bar{F}_0(\eta) := \begin{bmatrix} -\nabla\tilde{f}(X) - \mathbf{L}_0 X - Z \\ \mathbf{L}_0 X \end{bmatrix}$. Here $\mathbf{L}_0$ means that there does not exist any kind of attacks and the communication topology is still connected.

In this chapter, we mainly consider the scenarios where we allow attacker(s) to be able to switch topologies at will, until the attack(s) is (are) shut down and normal operation resumes. So it is convenient to model attack mode with a differential inclusion:

$$\dot{\eta} \in \overline{\text{con}} \bigcup_{\sigma\in\mathscr{Q}} \bar{F}_\sigma(\eta), \tag{6.7}$$

where $\bar{F}_\sigma(\eta) := \begin{bmatrix} -\nabla\tilde{f}(X) - \mathbf{L}_\sigma X - Z \\ \mathbf{L}_\sigma X \end{bmatrix}$. $\mathbf{L}_0$ and $\mathbf{L}_\sigma$ have zero-sum rows and zero-sum columns.

When the dynamics evolve according to (6.6) we say that the graph is in "good operation". When the dynamics evolve according to (6.7) we say that the graph is in "compromised operation". To find the optimal solution, an attack should neither occur too frequently nor last too long.

For simplicity, for any $t_2 > t_1 \geq 0$, in time interval $[t_1, t_2)$, let $T(t_2, t_1)$ denote the total activation time of compromised operation and $N(t_2, t_1)$ denote the number of switches between "good operation" and "compromised operation".

Assumption 6.2 There exist numbers $\rho \in [0, 1)$, $\delta \in \mathbb{R}_{>0}$, $N_0 \in \mathbb{Z}_{>0}$ and $T_0 \in \mathbb{R}_{>0}$ such that, for each $t_2 > t_1 \geq 0$, $T(t_2, t_1)$ satisfies the time-ratio constraint:

$$T(t_2, t_1) \leq T_0 + \rho(t_2 - t_1), \tag{6.8}$$

and $N(t_2, t_1)$ satisfies the average dwell-time constraint:

$$N(t_2, t_1) \leq N_0 + \delta(t_2 - t_1). \tag{6.9}$$

Note that when $\sigma = 0$, work [37] has shown that the equilibrium point $col(x_i^*, z_i^*)$ of (6.5) yields the optimal solution of problem (6.4). However, under networked attacks, the solutions of problem (6.3) may not achieve consensus since the graph may be unconnected and zero eigenvalue of Laplacian matrix may not be simple. This implies that the solution obtained under attacks may not be the optimal solution of problem (6.3). Moreover, the proposed approach may be used to deal with time-varying communications, such as those arising in a mobile sensor network where the links among nodes will come and go. In particular, disappearing links may be thought of as being like attacks. Also, our approach may be used to imitate controller/actuator failure cases.

6.4 Stability Analysis of the Algorithm Under Attacks

To easily explore convergence to the optimal solution under such attacks, we first make a coordinate transformation and give two lemmas. Then based on these lemmas, the main result will be presented.

Let (X^*, Z^*) be the equilibrium of (6.6). Define the following variables

$$\begin{aligned} Y &:= X - X^*, \quad \chi := [r\ \ R]^T Y \\ \overline{Z} &:= Z - Z^*, \quad \varsigma := [r\ \ R]^T \overline{Z} \end{aligned} \tag{6.10}$$

with $\chi := col(\chi_1, \chi_2) \in \mathbb{R}^{Nm}$, $\varsigma := col(\varsigma_1, \varsigma_2) \in \mathbb{R}^{Nm}$, $r := \hat{r} \otimes I_m$, $R := \hat{R} \otimes I_m$, where $\chi_1, \varsigma_1 \in \mathbb{R}^m$, $\chi_2, \varsigma_2 \in \mathbb{R}^{(N-1)m}$, $\hat{r} := \frac{\mathbf{1}_N}{\sqrt{N}}$, $\hat{r}^T\hat{R} = \mathbf{0}$, $\hat{R}^T\hat{R} = I_{N-1}$ and $\hat{R}\hat{R}^T = I_N - \frac{\mathbf{1}_N\mathbf{1}_N^T}{N}$. After a simple calculation, we have $[r\ \ R]\chi = Y$ and $[r\ \ R]\varsigma = \overline{Z}$. Here we define

$$\hbar(X) := \nabla \tilde{f}(X) - \nabla \tilde{f}(X^*). \tag{6.11}$$

Then it follows from (6.6), (6.7), (6.10) and (6.11) that, when $\sigma = 0$, we have

$$\dot{\xi} = F_0(\xi), \tag{6.12}$$

where $\xi := col(\chi_1, \chi_2, \varsigma_2)$ and

$$F_0(\xi) := \begin{bmatrix} -r^T\hbar(r\chi_1 + R\chi_2 + X^*) \\ -R^T\hbar(r\chi_1 + R\chi_2 + X^*) - R^T\mathbf{L}_0R\chi_2 - \varsigma_2 \\ R^T\mathbf{L}_0R\chi_2 \end{bmatrix}.$$

When $\sigma \in \mathcal{Q}$, we have

$$\dot{\xi} \in \overline{\text{con}} \bigcup_{\sigma \in \mathcal{Q}} F_\sigma(\xi), \tag{6.13}$$

where $F_\sigma(\xi) := \begin{bmatrix} -r^T \hbar(r\chi_1 + R\chi_2 + X^*) \\ -R^T \hbar(r\chi_1 + R\chi_2 + X^*) - R^T \mathbf{L}_\sigma R\chi_2 - \varsigma_2 \\ R^T \mathbf{L}_\sigma R\chi_2 \end{bmatrix}$.

To show that the optimal solution of (6.3) can be achieved exponentially when there are no attacks affecting the network, we consider the Lyapunov function candidate:

$$V_0(\xi) := \xi^T P_1 \xi \tag{6.14}$$

where $P_1 := \begin{bmatrix} \frac{\alpha}{2} I_m & \mathbf{0} & \mathbf{0} \\ * & \frac{\alpha}{2} I_{(N-1)m} & \frac{\varepsilon}{2} I_{(N-1)m} \\ * & * & \frac{\varXi J}{2} \end{bmatrix}$ with $\alpha, \varepsilon > 0$, $J := (R^T \mathbf{L}_0 R)^{-1}$, $\varXi := diag\{d_1, \ldots, d_{N-1}\} \otimes I_m > 0$.

Lemma 6.1 *Suppose Assumption 6.1 holds. Then, there exist numbers $\alpha, \varepsilon > 0$, and matrix $\varXi > 0$ such that $P_1 > 0$ and*

$$\langle \nabla V_0(\xi), F_0(\xi) \rangle \leq -\lambda_1 V_0(\xi) \tag{6.15}$$

hold, where $\lambda_1 := \frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P_1)}$, $Q_1 := \begin{bmatrix} \varLambda_1 & \boldsymbol{0} & \boldsymbol{0} \\ * & \varLambda_2 & \varLambda_3 \\ * & * & \frac{\varepsilon}{2} I_{(N-1)m} \end{bmatrix} > 0$ *with* $\varLambda_1 := \alpha\kappa I_m - \frac{\varepsilon \ell^2}{2} I_m$, $\varLambda_2 := \alpha\kappa I_{(N-1)m} + (\alpha - \varepsilon) J^{-1} - \frac{\varepsilon \ell^2}{2} I_{(N-1)m}$, $\varLambda_3 := \frac{\varepsilon}{2} J^{-1} - \frac{\varXi - \alpha I_{(N-1)m}}{2}$, $\kappa := \min\{\bar{\kappa}_1, ..., \bar{\kappa}_N\}$, $\ell := \max\{\ell_1, ..., \ell_N\}$.

Proof According to (6.12) and (6.14), we have

$$\begin{aligned} \langle \nabla V_0(\xi), F_0(\xi) \rangle = & -\alpha Y^T \hbar(r\chi_1 + R\chi_2 + X^*) - \alpha \chi_2^T R^T \mathbf{L}_0 R \chi_2 \\ & - \alpha \chi_2^T \varsigma_2 - \varepsilon \varsigma_2^T \varsigma_2 + \varepsilon \chi_2^T R^T \mathbf{L}_0 R \chi_2 + \varsigma_2^T \varXi \chi_2 \\ & - \varepsilon \varsigma_2^T R^T \hbar(r\chi_1 + R\chi_2 + X^*) - \varepsilon \varsigma_2^T R^T \mathbf{L}_0 R \chi_2. \end{aligned}$$

Since the local objective functions are strongly convex with constant $\bar{\kappa}_i$, we have

$$\begin{aligned} Y^T \hbar(r\chi_1 + R\chi_2 + X^*) = & (X - X^*)^T (\nabla \tilde{f}(X) - \nabla \tilde{f}(X^*)) \\ \geq & \kappa \chi^T \chi, \end{aligned} \tag{6.16}$$

where Y is defined in (6.10) and $\hbar(r\chi_1 + R\chi_2 + X^*)$ is defined in (6.11).

Applying Young's inequality gives that

$$-\varepsilon \varsigma_2^T R^T \hbar(r\chi_1 + R\chi_2 + X^*) \leq \frac{\varepsilon}{2} \varsigma_2^T \varsigma_2 + \frac{\varepsilon}{2} \ell^2 \chi^T \chi. \tag{6.17}$$

From (6.14), (6.16)–(6.17), it follows that

$$\begin{aligned}\langle \nabla V_0(\xi), F_0(\xi)\rangle \leq & -\alpha\kappa\chi^T\chi - (\alpha-\varepsilon)\chi_2^T R^T \mathbf{L}_0 R\chi_2 + \frac{\varepsilon}{2}\ell^2\chi^T\chi \\ & -\varepsilon\varsigma_2^T R^T \mathbf{L}_0 R\chi_2 - \frac{\varepsilon}{2}\varsigma_2^T\varsigma_2 + \chi_2^T \varXi\varsigma_2 - \alpha\chi_2^T\varsigma_2 \\ \leq & -\xi^T Q_1\xi.\end{aligned}$$

For any given α, $\varXi > 0$, there always exists a small enough constant $\varepsilon > 0$ such that $Q_1 > 0$. Therefore, we have

$$\langle \nabla V_0(\xi), F_0(\xi)\rangle \leq -\lambda_1 V_0(\xi).$$

The proof is completed. ☐

Next, to evaluate the bounds of the solutions of system (6.13) evolving during the persistent attacks, we consider the Lyapunov function candidate:

$$V_p(\xi) := \xi^T P_2\xi, \quad p \in \mathscr{Q}, \tag{6.18}$$

where $P_2 := \begin{bmatrix} \frac{\beta}{2}I_m & \mathbf{0} & \mathbf{0} \\ * & \frac{\beta}{2}I_{(N-1)m} & \frac{\tilde{\varepsilon}}{2}I_{(N-1)m} \\ * & * & \frac{\theta}{2}J \end{bmatrix}$ with $\beta, \tilde{\varepsilon}, \theta > 0$.

Lemma 6.2 *Suppose Assumption 6.1 holds. Then, there exist numbers* $\beta, \tilde{\varepsilon}, \theta > 0$ *such that* $P_2 > 0$ *and*

$$\langle \nabla V_p(\xi), F_p(\xi)\rangle \leq \lambda_2 V_p(\xi), \quad p \in \mathscr{Q} \tag{6.19}$$

hold, where $\lambda_2 := \max_{p\in\mathscr{Q}} \frac{|Q_{2p}|}{\lambda_{\min}(P_2)}$, $Q_{2p} := \begin{bmatrix} \varPi_1 & \boldsymbol{0} & \boldsymbol{0} \\ * & \varPi_2 & \varPi_3 \\ * & * & -\frac{\tilde{\varepsilon}}{2}I_{(N-1)m} \end{bmatrix}$ *with* $\varPi_1 := -\beta\kappa I_m + \frac{\tilde{\varepsilon}\ell^2}{2}I_m$, $\varPi_2 := -\beta\kappa I_{(N-1)m} + \frac{\tilde{\varepsilon}\ell^2}{2}I_{(N-1)m} - (\beta-\tilde{\varepsilon})R^T\boldsymbol{L}_p R$, $\varPi_3 := -\frac{\beta}{2}I_{(N-1)m} + \frac{\theta J - \tilde{\varepsilon}I_{(N-1)m}}{2}R^T\boldsymbol{L}_p R$.

Proof From (6.13) and (6.18), we have

$$\begin{aligned}\langle \nabla V_p(\xi), F_p(\xi)\rangle = & -\beta\chi_1^T r^T\hbar(r\chi_1 + R\chi_2 + X^*) \\ & -\beta\chi_2^T R^T\hbar(r\chi_1 + R\chi_2 + X^*) - \beta\chi_2^T R^T\mathbf{L}_p R\chi_2 \\ & -\beta\chi_2^T\varsigma_2 + \theta\varsigma_2^T J R^T\mathbf{L}_p R\chi_2 - \tilde{\epsilon}\varsigma_2^T\varsigma_2 \\ & +\tilde{\epsilon}\chi_2^T R^T\mathbf{L}_p R\chi_2 - \tilde{\epsilon}\varsigma_2^T R^T\hbar(r\chi_1 + R\chi_2 + X^*) \\ & -\tilde{\epsilon}\varsigma_2^T R^T\mathbf{L}_p R\chi_2 \\ \leq & |Q_{2p}|\xi^T\xi.\end{aligned} \tag{6.20}$$

Then, it follows from (6.18) and (6.20) that

$$\langle \nabla V_p(\xi), F_p(\xi) \rangle \leq \lambda_2 V_p(\xi).$$

The proof is completed. □

Before presenting the main result, we first give the definition of exponential convergence for switched systems.

Definition 6.1 The equilibrium of the switched system (6.5) under a class of switching signals S is said to be exponentially stable with rate of convergence $c > 0$ if there exists $M > 0$ such that, for any switching signal $\sigma \in S$ and any corresponding solution ξ, we have

$$|\xi(t)| \leq M|\xi(0)| \exp(-ct), \ \forall t \geq 0.$$

We now give the main result.

Theorem 6.1 *Suppose Assumption 6.1 holds. Choose parameters* $\alpha, \varepsilon, \beta, \tilde{\varepsilon}, \theta > 0$ *and matrix* $\Xi > 0$ *such that the conclusions of Lemmas 1 and 2 hold. Let* λ_1 *and* λ_2 *come from those lemmas and suppose* $\delta > 0$ *and* $\rho \in [0, 1)$ *satisfy*

$$(1-\rho)\lambda_1 - \rho\lambda_2 - \delta \ln(\mu) > 0 \tag{6.21}$$

with $\mu := \max\left\{\max_{\xi^T\xi=1} \frac{\xi^T P_2 \xi}{\xi^T P_1 \xi}, \max_{\xi^T\xi=1} \frac{\xi^T P_1 \xi}{\xi^T P_2 \xi}\right\}$.

Under these conditions, for the switched system (6.5) under the class of switching signals satisfying Assumption 6.2 with this δ *and* ρ*, the optimal solution* x^* *of problem (6.3) is exponentially stable with rate of convergence* $\frac{\lambda}{2}$*, where* $\lambda := \lambda_1 - \gamma$ *and* $\gamma := \delta \ln(\mu) + \rho(\lambda_1 + \lambda_2)$.

Remark 6.3 In (6.21), $(1-\rho)\lambda_1$ is used to measure the average rate of exponential decay of the stable Lyapunov functions due to the stable subsystems, while $\rho\lambda_2$ is used to measure their exponential growth due to the unstable subsystems and $\delta \ln(\mu)$ is used to measure their exponential growth due to the switches.

6.5 Stability Analysis

To show the convergence to the optimal solution of problem (6.3), we construct a hybrid system whose state consists of the state of the switched system, a switching signal p, and two auxiliary timers τ_1 and τ_2. The dynamics of the timers τ_1 and τ_2 are specifically designed to not only incorporate the effect of the average dwell-time constraint and time-ratio monitor, but also to be used to construct a Lyapunov function for the hybrid system.

6.5.1 A Construction of Hybrid System

To restrict the frequency of attacks, we introduce an automaton with auxiliary state $\tau_1 \in \mathbb{R}_{\geq 0}$, which is described by

$$\begin{aligned} \dot{\tau}_1 &\in [0, \delta], \quad \tau_1 \in [0, N_0], \\ \tau_1^+ &= \tau_1 - 1, \quad \tau_1 \in [1, N_0], \end{aligned} \tag{6.22}$$

where $\delta \in \mathbb{R}_{>0}$ and $N_0 \in \mathbb{Z}_{>0}$. In [8, Proposition 1.1], it has been proved that this automaton is able to generate each hybrid time domain E that satisfies the average dwell-time constraint

$$N(t,s) := j - i \leq N_0 + \delta(t-s), \ \forall (s,i), \ (t,j) \in E, \ (s,i) \preceq (t,j), \tag{6.23}$$

and for each hybrid time domain satisfying (6.23) there exists a solution of (6.22), having the said hybrid time domain (also see [32, 56–58]).

Recall that $T(t,s)$ denotes the total activation time of unstable modes $p \in \mathscr{Q}$ between times s and t, and $T(t,s)$ satisfies the bound in (6.8). From [97] we know that this kind of time-ratio constraint can be induced by a time-ratio monitor with an auxiliary state $\tau_2 \in \mathbb{R}_{\geq 0}$, which is described by

$$\dot{\tau}_2 \in [0, \rho] - \mathbb{I}_{\mathscr{Q}}(p), \quad \tau_2 \in [0, T_0], \tag{6.24}$$

where

$$\mathbb{I}_{\mathscr{Q}}(p) := \begin{cases} 1, & p \in \mathscr{Q} \\ 0, & otherwise. \end{cases} \tag{6.25}$$

In [67, Lemma 7], it has been proved that this monitor is able to generate each hybrid time domain E and signal $p : E \to \mathscr{P}$ that satisfy the time-ratio constraint

$$T(t,s) := \int_s^t \mathbb{I}_{\mathscr{Q}}(p(r, j(r)))dr \leq T_0 + \rho(t-s), \ \ (s,i), (t,j) \in E, \ \ (s,i) \preceq (t,j), \tag{6.26}$$

and for each hybrid time domain satisfying (6.26) there exists a solution of (6.24), having the said hybrid time domain.

Combining (6.22) and (6.24), we generalize (6.12) and (6.13) to a hybrid system with the state variable $\zeta := (\xi, p, \tau_1, \tau_2)$:

$$\begin{aligned} \dot{\zeta} &\in F(\zeta), \quad \zeta \in C, \\ \zeta^+ &\in G(\zeta), \quad \zeta \in D, \end{aligned} \tag{6.27}$$

where

$$F(\zeta) := \begin{cases} \begin{bmatrix} F_0(\xi) \\ \{0\} \\ [0, \delta] \\ [0, \rho] \end{bmatrix}, & \zeta \in C_1 \\ \begin{bmatrix} \overline{\text{con}} \bigcup_{p \in \mathcal{Q}} F_p(\xi) \\ \{0\} \\ [0, \delta] \\ [0, \rho] - 1 \end{bmatrix}, & \zeta \in C_2, \end{cases}$$

$$G(\zeta) := \begin{cases} \xi \\ \mathcal{P} \setminus \{p\} \\ \tau_1 - 1 \\ \tau_2, \end{cases}$$

$C_1 := \mathbb{R}^{(2N-1)m} \times \{0\} \times [0, N_0] \times [0, T_0]$,
$C_2 := \mathbb{R}^{(2N-1)m} \times \mathcal{Q} \times [0, N_0] \times [0, T_0]$,
$C := C_1 \cup C_2$,
$D := \mathbb{R}^{(2N-1)m} \times \mathcal{P} \times [1, N_0] \times [0, T_0]$.

6.5.2 An Explicit Lyapunov Function Proof of Convergence

Choose the following Lyapunov function for system (6.27):

$$V(\zeta) := V_p(\xi) \exp(\ln(\mu)\tau_1 + (\lambda_1 + \lambda_2)\tau_2), \tag{6.28}$$

where V_p, $p \in \mathcal{P}$ are from (6.14) and (6.18).

According to (6.21) and the definitions of λ, γ in Theorem 6.1, we have $\lambda > 0$. For all $\zeta \in C$, from Lemmas 1, 2 and (6.25), it holds that

(i) if $p \in \{0\}$,

$$\begin{aligned} \frac{\partial V(\zeta)}{\partial \zeta} \cdot F(\zeta) \leq & \frac{\partial V(\xi, p, \tau_1, \tau_2)}{\partial \xi} F_0(\xi) + \frac{\partial V(\xi, p, \tau_1, \tau_2)}{\partial \tau_1} \delta + \frac{\partial V(\xi, p, \tau_1, \tau_2)}{\partial \tau_2} \rho \\ \leq & (-\lambda_1 + \delta \ln(\mu) + \rho(\lambda_1 + \lambda_2)) V(\zeta) \\ \leq & -\lambda V(\zeta); \end{aligned} \tag{6.29}$$

(ii) if $p \in \mathcal{Q}$, similarly, we have

$$\begin{aligned}\frac{\partial V(\zeta)}{\partial \zeta} \cdot F(\zeta) \leq &(\lambda_2 + \delta \ln(\mu) + (\lambda_1 + \lambda_2)(\rho - 1))V(\zeta) \\ \leq &-\lambda V(\zeta).\end{aligned} \tag{6.30}$$

From μ given in Theorem 6.1, we obtain

$$V_p(\xi) \leq \mu V_q(\xi), \quad p, q \in \mathscr{P}. \tag{6.31}$$

For all $\zeta \in D$ and all $\zeta^+ \in G(\zeta)$, it follows from (6.27) and (6.31) that

$$\begin{aligned}V(\zeta^+) =&V_{p^+}(\xi^+) \exp(\ln(\mu)\tau_1^+ + (\lambda_1 + \lambda_2)\tau_2^+) \\ \leq&\mu V_p(\xi) \exp(\ln(\mu)\tau_1 + (\lambda_1 + \lambda_2)\tau_2 - \ln(\mu)) \\ =&V(\zeta).\end{aligned} \tag{6.32}$$

Combining (6.29)–(6.32), we obtain

$$V(\zeta(t, j)) \leq \exp(-\lambda t)V(\zeta(0, 0)), \ \forall (t, j) \in dom(\zeta). \tag{6.33}$$

From (6.14), (6.18) and (6.28), it follows that

$$\alpha_1 |\zeta|^2_{\mathbb{W}} \leq V(\zeta) \leq \alpha_2 |\zeta|^2_{\mathbb{W}} \exp(\varGamma), \tag{6.34}$$

where $\alpha_1 := \min_{i \in \{1,2\}} \lambda_{\min}(P_i)$, $\alpha_2 := \max_{i \in \{1,2\}} \lambda_{\max}(P_i)$, $\mathbb{W} := \{\mathbf{0}\} \times \mathscr{P} \times [0, N_0] \times [0, T_0]$, $\varGamma := N_0 \ln(\mu) + (\lambda_1 + \lambda_2)T_0$.

From the set $C \cup D$ we have $|\zeta|_{\mathbb{W}} = |\xi|$. Then we conclude the solution of the switched system satisfies

$$|\xi(t)| \leq \sqrt{\frac{\alpha_2 \exp(\varGamma)}{\alpha_1}} \exp\left(-\frac{\lambda}{2} t\right) |\xi(0)|. \tag{6.35}$$

So for each solution of the switched system with average dwell-time constraint and time-ratio constraint, there always exists a complete solution of hybrid system (6.27) corresponding to the solution of the switched system. Then from (6.35), we have the optimal solution x^* of (6.3) is exponentially stable for algorithm (6.5). The proof is completed. □

6.6 Example

Theorem 6.1 provides constraints on the frequency and duration of persistent attacks that guarantee convergence to the optimal solution of the problem (6.3). The larger these parameters are, the more active the attackers are allowed to be. It typically turns out that allowing a higher frequency for attacks is compatible with allowing smaller time ratios for attacks and vice versa. In this section, we explore a tradeoff between these parameters in the context of an example by exploiting the freedom in the choice of Lyapunov function.

In order to show how algorithm (6.5) achieves the optimal solution for the dynamical network when allowing networked attacks to occur, we consider the optimization problem (6.3) with

$$\begin{array}{ll} f_1(x) = 0.9(x+1)^2 & f_2(x) = (x-4)^2 \\ f_3(x) = 0.5x^2 - 1 & f_4(x) = 0.6x^2 + x \\ f_5(x) = (x+2)^2 & f_6(x) = 0.8x^2 + 2 \\ f_7(x) = (x-10)^2 & f_8(x) = \ln(e^{-0.1x} + e^{0.3x}) + 0.9x^2 \\ f_9(x) = \sin\frac{x}{2} + \frac{x^2}{2} & f_{10}(x) = \frac{x^2}{\sqrt{x^2+9}} + 0.6x^2. \end{array}$$

By calculation, we can obtain that $\kappa = 1, \ell = 2$. To solve this problem distributively, we assign each local objective function to each agent i. The original communication topology and the destroyed topologies by attacks are depicted in Fig. 6.1.

From Theorem 6.1, we must have that $\rho < \tilde{\rho} := \frac{\lambda_1 - \delta\ln(\mu)}{\lambda_1 + \lambda_2}$. Here for each δ, we search for $\tilde{\rho}$ to obtain the maximum allowed ρ by fixing P_1 and picking P_2 to minimize λ_2. Choose $\alpha = 4.46, \varepsilon = 1.81, \Xi = diag\{4.6, 4.6, 8, 8, 8, 8, 10, 10, 10\}$, $\beta = 0 : 0.1 : 80$, $\tilde{\varepsilon} = 0 : 0.05 : 40$ and $\theta = 0 : 0.4375 : 350$ and then the stability region is shown in Fig. 6.2.

We give one type of switching signal which is shown in Fig. 6.3 and this signal satisfies the average dwell-time and time-ratio constraints with $N_0 = 3, \delta = 0.0106$, $T_0 = 0.5$, $\rho = 0.004$. For simplicity, in dash line of Fig. 6.3, '0' means there does

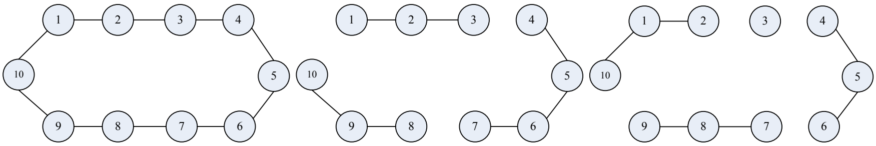
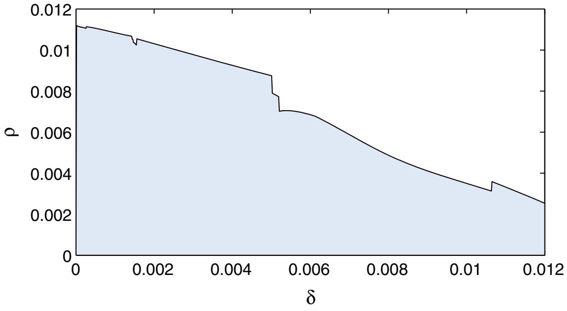


Fig. 6.1 The left is the original communication topology and the others are destroyed communication topologies by two types of attacks

Fig. 6.2 The stability region derived using Theorem 6.1 and different Lyapunov functions



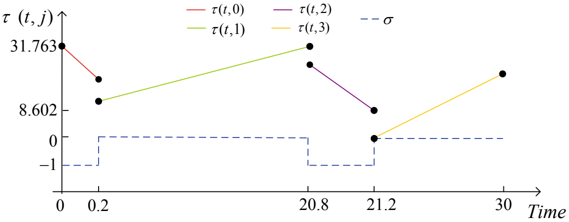


Fig. 6.3 Switching signal σ and combined timer τ

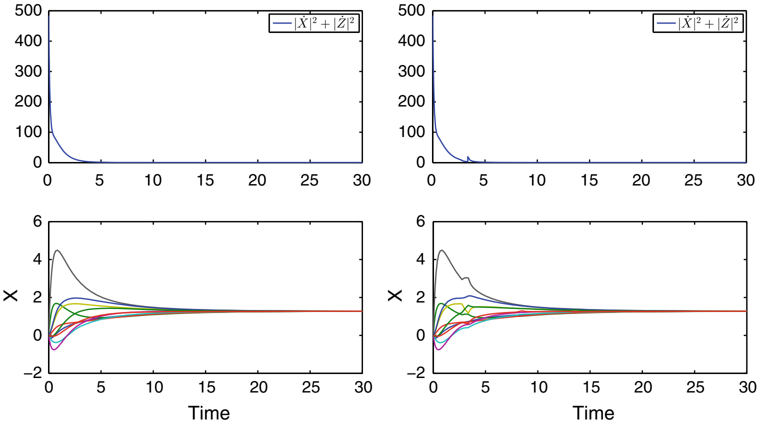


Fig. 6.4 The left two pictures denote the optimality condition and optimal solution without attacks. The right two pictures denote the optimality condition and optimal solution under attacks

not exist attacks and '-1' means attacks happened. According to (6.22) and (6.24), we plot the behavior of the combined timer $\tau := \ln(\mu)\tau_1 + (\lambda_1 + \lambda_2)\tau_2$ in Fig. 6.3. We see that τ decreases when attacks happen and jumps down when a switch occurs. Also, we see that it increases when there is no attack. Moreover, the simulation results are shown in Figs. 6.4 and 6.5. From Fig. 6.4, we can see that the solution under constrained attacks can converge to the optimal point ($x^* = x_1^* = x_2^* = \cdots = x_{10}^* = 1.2773$) which is the same as the optimal point generated without attacks and the corresponding optimal value is $g(x^*) = 109.5$. From Fig. 6.5 we observe that the optimal solution is exponentially convergent.

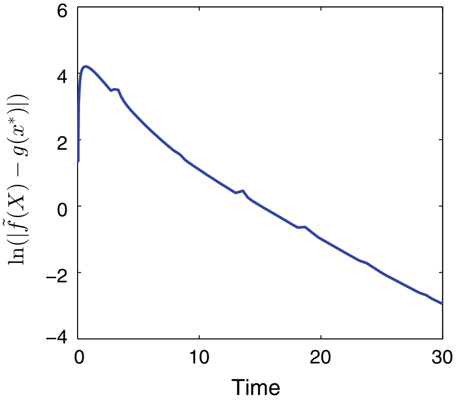


Fig. 6.5 The logarithmic curve of the error between the cost function and the optimal value under attacks

6.7 Conclusions

We have studied the effect of attacks on the ability of an algorithm to identify the optimal solution of a distributed optimization problem. To analyze this problem, we first modeled an algorithm as a switched algorithm and used differential inclusions to model attack modes. Then, to show exponential stability of the optimal solution, we modeled the switched algorithm as a hybrid dynamical system using the average dwell-time automaton and time-ratio monitor to constrain attacks, and thereby an explicit Lyapunov function is given.

Chapter 7
Distributed Robust Nash Equilibrium Seeking for Aggregative Games Under Persistent Attacks

7.1 Introduction

We explore robust distributed Nash equilibrium (NE) seeking for an aggregative game in which players have double-integrator dynamics and are influenced by the coexistence of unknown time-varying disturbances and unmodeled terms, and also are influenced by persistent attacks. Recently, some results on robust NE seeking have been obtained in [69, 98, 99]. However, these papers did not consider the effect of attacks on the communication topology. Persistent attacks may make the topology time varying and even unconnected during some time intervals (see [90, 101]). The authors in [90, 101] did not consider the effect of unknown terms on the convergence of the distributed algorithms. The unmodeled terms may create coupling among the players' dynamics and this makes the problem more challenging. Moreover, the disturbances and unmodeled dynamics may have devastating influences on the performance of the system.

To address the considered games, we regard unknown disturbances and unmodeled terms as an extended state. In addition, a reference signal is introduced to observe the integrator states and the corresponding error is used to design an estimator to estimate the unknown extended state. To obtain the aggregate of the decisions of all players, a distributed average consensus scheme is used while considering attacks on the communication topology. A differential inclusion is used to model attack modes (unstable modes). Furthermore, inspired by [22, 90], we model the closed-loop system as a hybrid system using an average dwell-time (ADT) automaton and a time-ratio monitor to model constraints on the frequency and duration of the attacks. Some work related to ADT automata and time-ratio constraints has been reported. For instance, the paper [32] proposed an ADT constraint to analyze stability for switched systems. The work [58] developed optimization-based methods for automatically verifying ADT properties of hybrid systems. The work [67] modeled switched systems as a hybrid system using an ADT automaton. Following these contributions, some results related to ADT automata or/and time-ratio constraints have been obtained in other similar contexts [8, 55–57, 78, 89, 90, 97].

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The main content of this chapter can be summarized as follows. First, for the unmodeled term, we need only the first-order derivative to be bounded, which is relaxed in comparison to [98]. Secondly, we assume the gradients of the cost functions are locally Lipschitz continuous, which is different from the global Lipschitz assumption on gradients used in [11, 69]. In addition, our aggregation function can be nonlinear and the cost functions can be non-quadratic, which is a more general setting than that considered in [11, 65, 102]. Thirdly, in the proposed distributed algorithm we consider the influence of persistent attacks on convergence to the NE. This is different from [11, 69, 70, 98–100, 102, 103], just to name a few, in which it is assumed that the communication topology is always connected. Moreover, to further analyze stability, the closed-loop system is modeled as a hybrid system. Then, a Lyapunov function is constructed and uniform (or uniform global) asymptotic stability is obtained.

7.2 Problem Formulation and Model Description

7.2.1 Game Formulation

We consider an aggregative game with N players and the communication topology among players is described by an undirected graph $\mathscr{G}$ with the node set $\mathscr{V}$. Each player $i \in \mathscr{V}$ aims to minimize its cost function $J_i : \mathbb{R}^{Nn} \to \mathbb{R}$, whose values are written as $J_i(x_i, x_{-i})$, where $x_{-i} := \mathrm{col}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$, by choosing a decision variable x_i. In addition, $J_i(x_i, x_{-i})$ is not known by other players $j\,(j \neq i)$. In general, player i faces the following optimization problem:

$$\min_{x_i \in \mathbb{R}^n} J_i(x_i, x_{-i}). \tag{7.1}$$

The following assumption is needed in the game.

Assumption 7.1 The cost function $x \mapsto J_i(x_i, x_{-i}), i \in \mathscr{V}$ is continuously differentiable.

We stack together the gradients of all cost functions to define the map $F_\nabla : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ as

$$F_\nabla(x) := \mathrm{col}(\nabla_{x_1} J_1(x_1, x_{-1}), \ldots, \nabla_{x_N} J_N(x_N, x_{-N})). \tag{7.2}$$

Assumption 7.2 $F_\nabla : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is strongly monotone with constant $m_f > 0$ in the following sense

$$(F_\nabla(x) - F_\nabla(y))^T(x-y) \geq m_f|x-y|^2, \ \forall x, y \in \mathbb{R}^{Nn}.$$

The NE of this game is defined as follows (see also [16, 60, 100]).

Definition 7.1 A strategy profile $x^* := \mathrm{col}(x_1^*, \ldots, x_N^*)$ is said to be a NE of the game (7.1) if

$$J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*), \ \forall x_i \in \mathbb{R}^n, \ i \in \mathcal{V}.$$

The NE of this game satisfies the following lemma (see [15] or [11]).

Lemma 7.1 *Under Assumptions 7.1 and 7.2,* $x^* := \mathrm{col}(x_1^*, \ldots, x_N^*)$ *is a unique NE of the game (7.1) if and only if*

$$\nabla_{x_i} J_i(x_i^*, x_{-i}^*) = \boldsymbol{0}, \ i \in \mathcal{V}.$$

The aggregation function of the game is $\sigma(x) := \frac{1}{N}\sum_{i=1}^{N} \varphi_i(x_i)$, where the continuously differentiable functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}^m, i \in \{1, \ldots, N\}$, constitute the local contribution to the aggregate and $x := \mathrm{col}(x_1, \ldots, x_N)$ is the strategy profile of the game.

In addition, the following assumptions are also needed.

Assumption 7.3 The aggregation function $x \mapsto \sigma(x)$ specifies the gradients of cost functions as $\nabla_{x_i} J_i(x_i, x_{-i}) = G_i^{\nabla}(x_i, \sigma(x))$ for functions $G_i^{\nabla} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, i \in \mathcal{V}$.

Assumption 7.4 For any compact sets $\mathbb{W}_1 \subset \mathbb{R}^n$ and $\mathbb{W}_2 \subset \mathbb{R}^m$ there exists a constant $\ell > 0$ such that

$$|G_i^{\nabla}(x_i, s_i) - G_i^{\nabla}(x_i, \bar{s}_i)| \leq \ell |s_i - \bar{s}_i|, \ \forall x_i \in \mathbb{W}_1, s_i, \bar{s}_i \in \mathbb{W}_2.$$

7.2.2 *Physical Model Description*

The dynamics of player $i \in \mathcal{V}$ are described as follows:

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i + d_i + g_i(x), \end{aligned} \tag{7.3}$$

where $x_i \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n$ are the position and velocity states of player i, respectively, u_i is the control input, d_i is an unknown time-varying external disturbance and $g_i(x)$ is the unmodeled term whose explicit expression is also unknown. Moreover, the disturbance d_i and unmodeled term g_i satisfy the following mild assumption.

Assumption 7.5 The functions $g_i, d_i, i \in \mathcal{V}$ are continuously differentiable, $\dot{d}_i$ is Lipschitz continuous, and $c_1, c_2 > 0$ are such that $|\dot{d}_i(t)| \leq c_1$ and $|\ddot{d}_i(t)| \leq c_2$ for almost all t.

Remark 7.2 The condition on d_i in Assumption 7.5 is also used in [34, 35, 98]. The assumption on g_i is different from [34, 35, 98] where it is assumed that the first-order and second-order derivatives with respect to x of g_i are bounded.

7.2.3 Networked Attack Descriptions

To obtain the aggregate of the decisions of all players, a distributed average consensus scheme will be used. The existing distributed algorithms for game problems assume that the topology among players is connected (see, for example, [11, 70, 88, 98] and references therein). Nevertheless, in many practical situations, ensuring that the communication network is always connected is impossible since the existence of networked attacks may lead to communication failures and may also result in divergence of the algorithms (see [90] and references therein). Therefore, it is meaningful to analyze the robustness of an NE seeking algorithm for an aggregative game to persistent attacks on the communication topology. To simplify, the set $\mathscr{Q} \subset \mathbb{Z}_{>0}$ is used to denote the types of possible persistent attacks and $\{0\}$ means that the topology does not encounter any attack.

Inspired by [90], for a distributed algorithm to succeed in finding a NE, an attack should neither occur too frequently nor last too long. A hybrid systems approach to the analysis of switching systems is useful when only switching signals from certain classes are allowed and the frequency of switching is limited. To limit how many attacks occur in a given time interval, the following clock state $\tau_1 \in \mathbb{R}_{\geq 0}$ is introduced:

$$\begin{aligned} \dot{\tau}_1 &\in [0, \delta], \quad \tau_1 \in [0, N_0], \\ \tau_1^+ &= \tau_1 - 1, \quad \tau_1 \in [1, N_0], \end{aligned} \tag{7.4}$$

where $\delta \in \mathbb{R}_{>0}$ and $N_0 \in \mathbb{R}_{\geq 1}$. The above model exactly captures the ADT constraint:

$$j - i \leq N_0 + \delta(t - s). \tag{7.5}$$

That is, every pair of hybrid times (s, i), (t, j) in a hybrid time domain E satisfies (7.5) if and only if E is the domain of some solution to (7.4) (see [8, Proposition 1.1] for a proof of this fact and also see [32, 56, 57]).

A hybrid system can be used to model the evolution of learning dynamics under persistent communication failures and different adversarial scenarios that may cause instability of the closed-loop system. To address this case the compact set $\mathscr{P}$ can be partitioned as $\mathscr{P} := \{0\} \cup \mathscr{Q}$, where the mode $q = 0$ characterizes the stable dynamics, while the modes $q \in \mathscr{Q}$ characterize the unstable dynamics. For such systems, good behavior of the solutions can be guaranteed as long as the amount of activation time of the unstable modes $\mathscr{Q}$ is bounded by a time-ratio constraint [97], which has the following form:

$$T(t,s) \le \frac{T_0}{\kappa} + \rho(t-s), \tag{7.6}$$

where $T(t,s)$ donotes the total activation time of unstable modes $p \in \mathscr{Q}$ between times s and t, $\rho \in [0,1)$, $T_0 \in \mathbb{R}_{\ge 0}$ and κ is from (9). From [97] we know that this kind of time-ratio constraint can be induced by a time-ratio monitor with an auxiliary state $\tau_2 \in \mathbb{R}_{\ge 0}$, which is described by

$$\dot{\tau}_2 \in [0, \kappa\rho] - \kappa \mathbb{I}_{\mathscr{Q}}(p), \;\; \tau_2 \in [0, T_0], \tag{7.7}$$

where $\mathbb{I}_{\mathscr{Q}}(p) := \begin{cases} 1, & p \in \mathscr{Q} \\ 0, & otherwise. \end{cases}$ This is to say that a hybrid time domain E satisfies (7.6) if and only if E is the domain of some solution to (7.7) (see [67, Lemma 7] for a proof of this fact).

7.3 Distributed Algorithm Design

We aim at designing the control inputs in (7.3) to achieve robust computation of a NE when such an NE exists. Let $w_i := d_i + g_i(x)$ and $\theta_i := \dot{d}_i(t)$, then the dynamics of player i with extended states can be written as

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i + w_i \\ \dot{w}_i &= \theta_i + \left(\frac{\partial g_i(x)}{\partial x}\right)^T v \\ \dot{\theta}_i &\in \mathbb{B}_{c_2}, \end{aligned} \tag{7.8}$$

where $w_i \in \mathbb{R}^n$, $\theta_i \in \mathbb{R}^n$, $v := \mathrm{col}\{v_1, \ldots, v_N\}$ and c_2 is given in Assumption 7.5.

Since different types of attacks may destroy different edges of the communication network, a new distributed algorithm based on a switching method is designed as follows:

$$u_i = -G_i^{\nabla}(x_i, s_i) - \widehat{w}_i - k v_i \tag{7.9a}$$

$$\dot{\widehat{v}}_i = u_i + \widehat{w}_i + (k_i + \kappa_i)(v_i - \widehat{v}_i) \tag{7.9b}$$

$$\dot{\widehat{w}}_i = k_i \kappa_i (v_i - \widehat{v}_i) + \beta_i sgn(v_i - \widehat{v}_i) \tag{7.9c}$$

$$\dot{s}_i = \kappa\Big(-(s_i - \varphi_i(x_i)) - \sum_{j \in \mathscr{N}_i} (a_{ij})_p (s_i - s_j) - z_i\Big) \tag{7.9d}$$

$$\dot{z}_i = \kappa \sum_{j \in \mathscr{N}_i} (a_{ij})_p (s_i - s_j), \;\; z \in \mathbb{A}, \tag{7.9e}$$

where $\widehat{v}_i \in \mathbb{R}^n$, $\widehat{w}_i \in \mathbb{R}^n$, $s_i \in \mathbb{R}^m$, $z_i \in \mathbb{R}^m$, $z := \mathrm{col}\{z_1, \ldots, z_N\}$, $z \in \mathbb{A} := \{z \in \mathbb{R}^{Nm} | \sum_{i=1}^{N} z_i = \mathbf{0}\}$. It can be shown that $\mathbb{A}$ is forward invariant. $(a_{ij})_p \in \{0, 1\}$ and $p : [t_0, +\infty) \to \mathscr{P}$ is a piecewise constant function. The positive parameters k, k_i, κ_i, κ are to be determined later.

Remark 7.3 In this algorithm, $\widehat{v}_i$ could be regarded as a reference signal for v_i and the corresponding error signal is used to obtain an estimation $\widehat{w}_i$ of the unknown extended state w_i. $G_i^{\nabla}(x_i, s_i)$ is as an approximation of gradients for the presented game. s_i is for the estimation of the aggregative function and z_i is as an auxiliary variable for ensuring exact estimation.

Define the following error signals:

$$\begin{aligned} &\zeta_{vi} := v_i - \widehat{v}_i,\ \widetilde{w}_i := w_i - \widehat{w}_i,\ \chi_i := x_i - x_i^*,\\ &e_{si} := s_i - s_i^*,\ e_{zi} := z_i - z_i^*, \end{aligned} \tag{7.10}$$

where x_i^* is the component of the NE for player i, $s_i^* := \frac{1}{N}\sum_{i=1}^{N} \varphi_i(x_i)$, $z_i^* := \varphi_i(x_i) - s_i^*$.

It follows from (7.10) and the definition of $\mathbb{A}$ given in (7.9e) that

$$\sum_{i=1}^{N} e_{zi} = \sum_{i=1}^{N}(z_i - z_i^*) = \sum_{i=1}^{N}(z_i - (\varphi_i(x_i) - s_i^*)) = \sum_{i=1}^{N} z_i = 0. \tag{7.11}$$

From (7.8)–(7.11), we obtain the following system:

$$\begin{aligned} \dot{\zeta}_{vi} &= -(k_i + \kappa_i)\zeta_{vi} + \widetilde{w}_i\\ \dot{\widetilde{w}}_i &= -k_i\kappa_i\zeta_{vi} - \beta_i sgn(\zeta_{vi}) + \theta_i + \tilde{g}_i\\ \dot{v}_i &= -kv_i - G_i^{\nabla}(\chi_i + x_i^*, e_{si} + s_i^*) + \widetilde{w}_i\\ \dot{\chi}_i &= v_i\\ \dot{\theta}_i &\in \mathbb{B}_{c_2}\\ \dot{e}_{si} &= \kappa(-e_{si} - \sum_{j\in\mathscr{N}_i}(a_{ij})_p(e_{si} - e_{sj}) - e_{zi}) - \psi_i\\ \dot{e}_{zi} &= \kappa\sum_{j\in\mathscr{N}_i}(a_{ij})_p(e_{si} - e_{sj}) - \psi_{xi} + \psi_i,\ e_z \in \mathbb{A}, \end{aligned} \tag{7.12}$$

where $p \in \mathscr{P}$, $\chi := \mathrm{col}(\chi_1, \ldots, \chi_N)$, $\tilde{g}_i := \left(\frac{\partial g_i(\chi + x^*)}{\partial(\chi + x^*)}\right)^T v$, $\psi_i := \frac{1}{N}\sum_{i=1}^{N}\left(\left(\frac{\partial\varphi_i(\chi_i + x_i^*)}{\partial(\chi_i + x_i^*)}\right)^T v_i\right)$, $\psi_{xi} := \left(\frac{\partial\varphi_i(\chi_i + x_i^*)}{\partial(\chi_i + x_i^*)}\right)^T v_i$, $e_z := \mathrm{col}\{e_{z1}, \ldots, e_{zN}\}$.

The algorithm corresponding to (7.12) is distributed since each player exchanges information only with its neighbors,

Denote $\zeta_v := \mathrm{col}(\zeta_{v1}, \ldots, \zeta_{vN})$, $\widetilde{w} := \mathrm{col}(\widetilde{w}_1, \ldots, \widetilde{w}_N)$, $\theta := \mathrm{col}(\theta_1, \ldots, \theta_N)$, $e_s := \mathrm{col}(e_{s1}, \ldots, e_{sN})$, $s^* := \mathrm{col}(s_1^*, \ldots, s_N^*)$, $\tilde{g} := \mathrm{col}(\tilde{g}_1, \ldots, \tilde{g}_N)$, $\phi_1(\chi +$

$x^*, v) := \mathrm{col}(\psi_1, \ldots, \psi_N)$, $\phi_2(\chi + x^*, v) := \mathrm{col}(\psi_{x1}, \ldots, \psi_{xN})$, $k_c := \mathrm{diag}\{k_i\}$, $\kappa_c := \mathrm{diag}\{\kappa_i\}$, $\beta_c := \mathrm{diag}\{\beta_i\}$, $\mathbf{L}_p := L_p \otimes I_m$, $\kappa_p := (k_c + \kappa_c) \otimes I_n$, $\kappa_I := (k_c \kappa_c) \otimes I_n$ and $\beta_I := \beta_c \otimes I_n$. Then we can rewrite (7.12) in a Kronecker product form:

$$\begin{aligned}
\dot{\zeta}_v &= -\kappa_p \zeta_v + \widetilde{w} \\
\dot{\widetilde{w}} &= -\kappa_I \zeta_v - \beta_I sgn(\zeta_v) + \theta + \tilde{g} \\
\dot{v} &= -kv - G^{\nabla}(\chi + x^*, e_s + s^*) + \widetilde{w} \\
\dot{\chi} &= v \\
\dot{\theta} &\in \mathbb{B}^N_{c_2} \\
\dot{e}_s &= \kappa(-e_s - \mathbf{L}_p e_s - e_z) - \phi_1(\chi + x^*, v) \\
\dot{e}_z &= \kappa \mathbf{L}_p e_s - \phi_2(\chi + x^*, v) + \phi_1(\chi + x^*, v),\ e_z \in \mathbb{A}.
\end{aligned} \tag{7.13}$$

The right-hand side of system (7.13) is discontinuous. We will use Krasovskii regularization to transform it into a differential inclusion since this regularization accurately predicts the effect of small state perturbations on solutions to a differential equation with discontinuous right-hand side (see [24]). Inspired by [24], the work in [22, Chap. 4] applied this idea to hybrid systems, which are used in the sequel.

To further obtain the complete closed-loop system, we perform the following orthogonal transformation:

$$\vartheta := [r \ \ R]^T e_s, \ \ \xi := [r \ \ R]^T e_z \tag{7.14}$$

with $\vartheta := \mathrm{col}(\vartheta_1, \vartheta_2) \in \mathbb{R}^{Nm}$, $\xi := \mathrm{col}(\xi_1, \xi_2) \in \mathbb{R}^{Nm}$, $r := \hat{r} \otimes I_m$, $R := \hat{R} \otimes I_m$, where $\vartheta_1, \xi_1 \in \mathbb{R}^m$, $\vartheta_2, \xi_2 \in \mathbb{R}^{(N-1)m}$, $\hat{r} := \frac{\mathbf{1}_N}{\sqrt{N}}$, $\hat{r}^T \hat{R} = \mathbf{0}$, $\hat{R}^T \hat{R} = I_{N-1}$ and $\hat{R}\hat{R}^T = I_N - \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}$. After a simple calculation, we have $[r \ \ R]\vartheta = e_s$ and $[r \ \ R]\xi = e_z$.

Then, combining (7.13) and (7.14), we have

$$\begin{aligned}
\dot{\zeta}_v &= -\kappa_p \zeta_v + \widetilde{w} \\
\dot{\widetilde{w}} &\in -\kappa_I \zeta_v - \beta_I SGN(\zeta_v) + \theta + \tilde{g} \\
\dot{v} &= -kv - G^{\nabla}(\chi + x^*, r\vartheta_1 + R\vartheta_2 + s^*) + \widetilde{w}
\end{aligned}$$

$$\begin{aligned}
\dot{\chi} &= v \\
\dot{\theta} &\in \mathbb{B}^N_{c_2} \\
\dot{\vartheta}_1 &= -\kappa \vartheta_1 - r^T \phi_1(\chi + x^*, v) \\
\dot{\vartheta}_2 &= \kappa(-\vartheta_2 - \xi_2 - R^T \mathbf{L}_p R \vartheta_2) - R^T \phi_1(\chi + x^*, v) \\
\dot{\xi}_2 &= \kappa R^T \mathbf{L}_p R \vartheta_2 - R^T \phi_2(\chi + x^*, v) + R^T \phi_1(\chi + x^*, v) \\
\xi_1 &= 0.
\end{aligned} \tag{7.15}$$

In (7.15) we have used the constraint $e_z \in \mathbb{A}$ and from this constraint we can obtain that $\xi_1 = 0$. Clearly, (7.15) is equivalent to (7.13). Thus, we can analyze stability of (7.15) to show stability of (7.13).

Combining (7.4) and (7.7), we can model (7.15) as a hybrid system with the state variable $\zeta := \mathrm{col}(\nu, \widetilde{\sigma}, \tau_1, \tau_2, y)$:

$$\begin{aligned} \dot{\zeta} &\in F(\zeta), \quad \zeta \in C \\ \zeta^+ &\in G(\zeta), \quad \zeta \in D, \end{aligned} \tag{7.16}$$

where $\nu := \mathrm{col}(\zeta_v, \widetilde{w}, v, \chi, \theta)$, $y := \mathrm{col}(\vartheta_1, \vartheta_2, \xi_2)$,

$$F(\zeta) := \begin{cases} \begin{bmatrix} f(\nu, \vartheta_1, \vartheta_2) \\ \{0\} \\ [0, \delta] \\ [0, \kappa\rho] \\ \kappa F_0(y, \chi, v) \end{bmatrix}, & \zeta \in C_1 \\ \begin{bmatrix} f(\nu, \vartheta_1, \vartheta_2) \\ \{0\} \\ [0, \delta] \\ [0, \kappa\rho] - \kappa \\ \overline{\mathrm{co}} \bigcup\limits_{p \in \mathscr{Q}} \kappa F_p(y, \chi, v) \end{bmatrix}, & \zeta \in C_2, \end{cases}$$

$$f(\nu, \vartheta_1, \vartheta_2) := \begin{bmatrix} -\kappa_p \zeta_v + \widetilde{w} \\ -\kappa_I \zeta_v - \beta_I SGN(\zeta_v) + \theta + \tilde{g} \\ -kv - G^{\nabla}(\chi + x^*, r\vartheta_1 + R\vartheta_2 + s^*) + \widetilde{w} \\ v \\ \mathbb{B}^N_{c_2} \end{bmatrix},$$

$$F_0(y, \chi, v) := \begin{bmatrix} -\vartheta_1 - \frac{1}{\kappa} r^T \phi_1(\chi + x^*, v) \\ -\vartheta_2 - \xi_2 - R^T \mathbf{L}_0 R \vartheta_2 - \frac{1}{\kappa} R^T \phi_1(\chi + x^*, v) \\ R^T \mathbf{L}_0 R \vartheta_2 - R^T \phi_2(\chi + x^*, v) + \frac{1}{\kappa} R^T \phi_1(\chi + x^*, v) \end{bmatrix},$$

$$F_p(y, \chi, v) := \begin{bmatrix} -\vartheta_1 - \frac{1}{\kappa} r^T \phi_1(\chi + x^*, v) \\ -\vartheta_2 - \xi_2 - R^T \mathbf{L}_p R \vartheta_2 - \frac{1}{\kappa} R^T \phi_1(\chi + x^*, v) \\ R^T \mathbf{L}_p R \vartheta_2 - R^T \phi_2(\chi + x^*, v) + \frac{1}{\kappa} R^T \phi_1(\chi + x^*, v) \end{bmatrix},$$

$$G(\nu, \widetilde{\sigma}, \tau_1, \tau_2, y) := \mathrm{col}(\nu, \mathscr{P} \setminus \{\widetilde{\sigma}\}, \tau_1 - 1, \tau_2, y), \tag{7.17}$$

and

$$\begin{aligned} C_1 &:= (\mathbb{R}^{4Nn} \times \mathbb{B}_{c_1}^N) \times \{0\} \times [0, N_0] \times [0, T_0] \times \mathbb{R}^{(2N-1)m}, \\ C_2 &:= (\mathbb{R}^{4Nn} \times \mathbb{B}_{c_1}^N) \times \mathscr{Q} \times [0, N_0] \times [0, T_0] \times \mathbb{R}^{(2N-1)m}, \\ C &:= C_1 \cup C_2, \\ D &:= (\mathbb{R}^{4Nn} \times \mathbb{B}_{c_1}^N) \times \mathscr{P} \times [1, N_0] \times [0, T_0] \times \mathbb{R}^{(2N-1)m}. \end{aligned}$$

Note that $\mathscr{P} := \{0\} \cup \mathscr{Q}$ while c_1 is given in Assumption 7.5, and κ_p, κ_I, β_I, k and κ are control gains. The auxiliary timers τ_1 and τ_2 are specifically designed to incorporate the effect of the ADT constraint and the time-ratio monitor. $\widetilde{\sigma}$ is a logic variable used to identify whether an attack happens. Here, $\mathbf{L}_0$ means there is no attack on the network and the communication topology is connected. $\mathbf{L}_p$ means that an attack is happening. $\mathbf{L}_0$ and $\mathbf{L}_p$ both have zero-sum rows and zero-sum columns.

In next section we will show how to select the gains within the control scheme to guarantee convergence to the NE.

7.4 Stability of the Hybrid Algorithm

In Sect. 4, we modeled a robust distributed NE seeking algorithm under persistent attacks as a hybrid system. In this section, we will give some conditions on the hybrid system and the control gains such that the compact set

$$\mathbb{W} := (\{\mathbf{0}\} \times \mathbb{B}_{c_1}^N) \times \mathscr{P} \times [0, N_0] \times [0, T_0] \times \{\mathbf{0}\}.$$

is uniformly asyptotically stable. If this set is uniformly asymptotically stable then, from Lemma 7.1, Assumption 7.3, (7.10) and (7.16), we know that the system consisting of (7.8) and (7.9) converges to the NE of the game (7.1).

To help deduce stability condition for the algorithm, two lemmas will be given. They show, respectively, that the NE can be found when there are no attacks against the network, and that bounds on the solutions during the persistent attacks can be evaluated. Choose the following Lyapunov function candidates:

$$\begin{aligned} V_0(y) &:= y^T P_0 y \\ V_p(y) &:= y^T P_1 y, \;\; p \in \mathscr{Q}, \end{aligned} \tag{7.18}$$

where $P_0 := \begin{bmatrix} \frac{\gamma_0}{2} I_m & \mathbf{0} & \mathbf{0} \\ * & \frac{\gamma_0 I_{(N-1)m}}{2} & \frac{\gamma_1 I_{(N-1)m}}{2} \\ * & * & \frac{\gamma_0}{2} J \end{bmatrix}$, $P_1 := \begin{bmatrix} \frac{\gamma_2}{2} I_m & \mathbf{0} & \mathbf{0} \\ * & \frac{\gamma_2 I_{(N-1)m}}{2} & \frac{\gamma_3 I_{(N-1)m}}{2} \\ * & * & \frac{\gamma_4}{2} J \end{bmatrix}$ with $J := (R^T \mathbf{L}_0 R)^{-1}$, $\gamma_i > 0, i \in \{0, \ldots, 4\}$.

Lemma 7.4 *Suppose Assumption 7.5 holds. There exist constants* $\gamma_0, \gamma_1 > 0$ *such that* $P_0 > 0$ *and*

$$\langle \nabla V_0(y), F_0(y, \chi, \upsilon) \rangle \leq -\lambda_1 V_0(y) + \frac{\varrho_1}{\kappa}$$

hold for all values of y, χ *and* ν, *where* $\lambda_1 := \frac{\lambda_{\min}(Q_0)}{\lambda_{\max}(P_0)}$, $Q_0 := \begin{bmatrix} \gamma_0 I_m & \boldsymbol{0} & \boldsymbol{0} \\ * & \Lambda_1 & \frac{\gamma_1}{2} J^{-1} \\ * & * & \frac{\gamma_1 I_{(N-1)m}}{2} \end{bmatrix} > 0$, $\Lambda_1 := \gamma_0 I_{(N-1)m} - \frac{\gamma_1 I_{(N-1)m}}{2} + (\gamma_0 - \gamma_1) J^{-1}$, $\varrho_1 := (2\gamma_0 + 2\gamma_1 + \frac{\gamma_0}{\lambda_2})|y||\phi_1| + (\gamma_1 + \frac{\gamma_0}{\lambda_2})|y||\phi_2|$, $\phi_1 := \phi_1(\chi + x^*, \upsilon)$, $\phi_2 := \phi_2(\chi + x^*, \upsilon)$.

Lemma 7.5 *Suppose Assumption 7.5 holds. There exist constants* $\gamma_2, \gamma_3, \gamma_4 > 0$ *such that* $P_1 > 0$ *and*

$$\langle \nabla V_p(y), F_p(y, \chi, \upsilon) \rangle \leq \lambda_2 V_p(y) + \frac{\varrho_2}{\kappa}, \ p \in \mathcal{Q}$$

hold for all values of y, χ *and* ν, *where* $\lambda_2 := \max\limits_{p \in \mathcal{Q}} \frac{|Q_{1p}|}{\lambda_{\min}(P_1)}$, $Q_{1p} := \begin{bmatrix} -\gamma_2 I_m & \boldsymbol{0} & \boldsymbol{0} \\ * & \Xi_1 & \Xi_2 \\ * & * & -\gamma_3 I_{(N-1)m} \end{bmatrix}$, $\Xi_1 := -\gamma_2 I_{(N-1)m} - (\gamma_2 - \gamma_3) R^T \boldsymbol{L}_p R$, $\Xi_2 := -\frac{(\gamma_2 + \gamma_3) I_{(N-1)m}}{2} + \frac{\gamma_4 J - \gamma_3 I_{(N-1)m}}{2} R^T \boldsymbol{L}_p R$, $\varrho_2 := (2\gamma_2 + 2\gamma_3 + \frac{\gamma_4}{\lambda_2})|y||\phi_1| + (\gamma_3 + \frac{\gamma_4}{\lambda_2})|y||\phi_2|$.

Remark 7.6 We emphasize that the parameters γ_i, $i \in \{0, \ldots, 4\}$, do not depend on the value of x^*. However, the expressions ρ_1 and ρ_2 in the two lemmas do depend on x^*. The proofs of Lemmas 7.4 and 7.5 are similar to the proofs of Lemmas 6.1 and 6.2, so we omit the proofs here.

Next, we give the main result.

Theorem 7.7 *Suppose Assumptions 7.1–7.5 hold. Choose parameters* $\gamma_i > 0, i \in \{0, \ldots, 4\}$ *such that the conclusions of Lemmas 7.4 and 7.5 hold. Let* λ_1 *and* λ_2 *come from those lemmas and suppose* $\kappa, \delta > 0$ *and* $\rho \in [0, 1)$ *satisfy*

$$(1 - \rho)\lambda_1 - \rho\lambda_2 - \frac{\delta \ln(\mu)}{\kappa} > 0 \tag{7.19}$$

with $\mu := \max\left\{ \max\limits_{y^T y = 1} \frac{y^T P_1 y}{y^T P_0 y}, \max\limits_{y^T y = 1} \frac{y^T P_0 y}{y^T P_1 y} \right\}$. *Then there exist constants* $k^*, \beta^* > 0$ *such that for all* $k > k^*$ *and* $\beta_i > \beta^*$, *the set* $\mathbb{W}$ *is uniformly asymptotically stable for the hybrid system (7.16).*

The proof is presented in Sect. 7.5.

Remark 7.8 The local stability of a NE can also be found in [16, 103]. The estimate of the basin of attraction can be obtained by the constructed Lyapunov function in Sect. 7.5. In addition, we use the ADT and time-ratio constrained parameters ρ and δ to analyze the behavior under attacks. The larger these parameters, the more active attackers are allowed to be. To guarantee the convergence to the NE, we use inequality (7.19) to constrain these parameters. From Theorem 7.7, we must have that $\rho < \frac{\lambda_1 - \frac{\delta \ln(\mu)}{\kappa}}{\lambda_1 + \lambda_2}$. For each δ, one can use a grid search method to obtain the maximum allowed ρ by fixing P_0 (i.e., γ_0, γ_1) and picking P_1 (i.e., $\gamma_2, \gamma_3, \gamma_4$) to minimize λ_2. About these parameters, in our previous work [90], we have used a Lyapunov function approach to maximize the time-ratio parameter by fixing the ADT parameter and then gave a tradeoff curve between parameters ρ and δ.

In Theorem 7.7, we assume that the gradients of cost functions of all players are locally Lipschitz continuous and the aggregation function is continuously differentiable, and uniform asymptotic stability is obtained. If we consider the following standard Assumptions 7.6 and 7.7 that are used in [11, 65, 98], uniform global asymptotic stability can be obtained.

Assumption 7.6 $s_i \mapsto G_i^{\nabla}(x_i, s_i)$ is Lipschitz continuous in s_i, uniform in x_i.

Assumption 7.7 $\tilde{F}$ is strongly monotone and Lipschitz continuous.

Assumption 7.8 The functions $\varphi_i, i \in \mathscr{V}$ are continuously differentiable and $x \mapsto \frac{\partial g_i(x)}{\partial x}$, $x_i \mapsto \frac{\partial \varphi_i(x_i)}{\partial x_i}$ are bounded.

Theorem 7.9 *Suppose Assumptions 7.1, 7.3, 7.5 and 7.6–7.8 hold. Choose parameters* $\gamma_i > 0, i \in \{0, \ldots, 4\}$ *such that the conclusions of Lemmas 7.4 and 7.5 hold. Let* λ_1 *and* λ_2 *come from those lemmas and suppose* $\kappa, \delta > 0$ *and* $\rho \in [0, 1)$ *satisfy inequality (7.19). Then there exist constants* $k^*, \beta^* > 0$ *such that for all* $k > k^*$ *and* $\beta_i > \beta^*$*, the set* $\mathbb{W}$ *is uniformly globally asymptotically stable for the hybrid system (7.16).*

The proof is a simple adaptation of the proof of Theorem 7.7, and so is omitted.

7.5 Proofs of the Main Theorems

Before presenting the proof of Theorem 7.7, we will first give one lemma which can be found in [3, 36, 82]. Moreover, some definitions about Clarke's generalized gradient, generalized directional derivative and regular function can be found in [10]. Consider the following differential inclusion:

$$\dot{x} \in F(x), \tag{7.20}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is OSC and LB.

The following lemma can be found in [3, 36, 82].

Lemma 7.10 *Let $x(\cdot)$ be a solution to (7.20) and $V : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz and in addition, regular function. Then $t \mapsto V(x(t))$ is absolutely continuous, $\frac{d}{dt}V(x(t))$ exists for almost every t and, where it exists, it satisfies:*

$$\frac{d}{dt}V(x(t)) \le \min_{p\in\partial V(x(t))} \max_{q\in F(x(t))} p^T q.$$

We now give the proof of Theorem 7.7.

Proof of Theorem 7.7 Choose the following Lyapunov function:

$$U(\zeta) := W(\nu) + \frac{c_v}{\kappa} V(y_c), \tag{7.21}$$

where $y_c := \mathrm{col}(\widetilde{\sigma}, \tau_1, \tau_2, y)$, y, ν are from (7.16), $V_{\widetilde{\sigma}}, \widetilde{\sigma} \in \mathscr{P} := \{0\} \cup \mathscr{Q}$ are from (7.18), $\kappa, c_v > 0$

$$\begin{aligned} V(y_c) &:= V_{\widetilde{\sigma}}(y)\exp(\ln(\mu)\tau_1 + (\lambda_1+\lambda_2)\tau_2) \\ W(\nu) &:= \frac{1}{2}\zeta_v^T\zeta_v + \frac{1}{2}\widetilde{w}^T Q\widetilde{w} - \alpha\zeta_v^T\widetilde{w} + \sum_{i=1}^N m_i|\zeta_{vi}| \\ &\quad - \theta^T Q\zeta_v + \frac{1}{2}v^T v + \frac{kb_k}{2}\chi^T\chi + b_k\chi^T v \end{aligned} \tag{7.22}$$

with $Q{:=}\Pi_1\Pi_2$, $\Pi_1{:=}(\alpha(k_c+\kappa)+I_N)\otimes I_n$, $\Pi_2{:=}\kappa_I^{-1}$, $m_i := \frac{(\alpha(k_i+\kappa_i)+1)\beta_i}{k_i\kappa_i}$, $i \in \mathscr{V}$, $0 < b_k < k$, and

$$\alpha^2 < \min\left\{\frac{b}{k_i\kappa_i}, \frac{(b-1)^2}{(k_i+\kappa_i)^2}\right\},\ b > 1, \alpha > 0,\ i \in \mathscr{V}. \tag{7.23}$$

Denote $M := \mathrm{diag}\{m_i\} \otimes I_n$ and note that κ_I is from (7.13) and Q, M are diagonal matrices. From (7.23), one can derive $\alpha(k_i+\kappa_i)+1 \le b$, $\forall i \in \mathscr{V}$. Applying the fact that $c_1\omega_1^2 + c_2\omega_1\omega_2 + c_3\omega_2^2$ is a positive definite function with respect to ω_1, ω_2 under the conditions $c_2^2 - 4c_1c_3 < 0$ and $c_1, c_3 > 0$, we conclude that $W(\nu) = 0$ if $\nu = 0$ and $W(\nu) > 0$, otherwise, and it is radially unbounded. In addition, $\sum_{i=1}^N m_i|\zeta_{vi}|$ is Lipschitz continuous. From Clarke's generalized gradient and (7.21), we have

$$\partial U(\zeta) = \begin{bmatrix} \zeta_v - \alpha\widetilde{w} + M^T SGN(\zeta_v) - Q^T\theta \\ -\alpha\zeta_v + Q^T\widetilde{w} \\ v + b_k\chi \\ kb_k\chi + b_k v \\ -\zeta_v^T Q^T \\ \frac{c_v}{\kappa}\frac{\partial V(y_c)}{\partial\widetilde{\sigma}} \\ \frac{c_v}{\kappa} V_{\widetilde{\sigma}}(y)\ln(\mu)\exp(\ln(\mu)\tau_1 + (\lambda_1+\lambda_2)\tau_2) \\ \frac{c_v}{\kappa} V_{\widetilde{\sigma}}(y)(\lambda_1+\lambda_2)\exp(\ln(\mu)\tau_1 + (\lambda_1+\lambda_2)\tau_2) \\ \frac{c_v}{\kappa}\frac{\partial V_{\widetilde{\sigma}}(y)}{\partial y}\exp(\ln(\mu)\tau_1 + (\lambda_1+\lambda_2)\tau_2) \end{bmatrix}.$$

For all $\zeta \in C$, from Lemmas 7.4, 7.5, (7.4), (7.7) and (7.16), it holds that

(i) if $\widetilde{\sigma} \in \{0\}$, for all $\zeta \in \mathbb{B}_{\bar{r}}$ we have

$$\begin{aligned} \min_{\upsilon\in\partial U(\zeta)}\max_{q\in F(\zeta)} \upsilon^T q \le{}& -\zeta_v^T\kappa_p\zeta_v + \alpha\widetilde{w}^T\kappa_p\zeta_v - SGN(\zeta_v)^T M\kappa_p\zeta_v + \zeta_v^T Q^T\kappa_p\theta + \zeta_v^T\widetilde{w} \\ & - \alpha\widetilde{w}^T\widetilde{w} - \widetilde{w}^T Q^T\theta + \alpha\zeta_v^T\kappa_I\zeta_v + \alpha\zeta_v^T\beta_I SGN(\zeta) - \alpha\zeta_v^T\theta \\ & - \alpha\zeta_v^T\tilde{g} - \widetilde{w}^T Q\kappa_I\zeta_v + \widetilde{w}^T Q\theta + \widetilde{w}^T Q\tilde{g} - kv^T v \\ & - v^T G^{\nabla}(\chi + x^*, r\vartheta_1 + R\vartheta_2 + s^*) + \widetilde{w}^T v - b_k k\chi^T v \\ & - b_k\chi^T G^{\nabla}(\chi + x^*, r\vartheta_1 + R\vartheta_2 + s^*) + b_k\widetilde{w}^T\chi + b_k k\chi^T v \\ & + b_k v^T v + |Q|c_2|\zeta_v| - (\lambda_1 - \frac{\delta\ln(\mu)}{\kappa} - (\lambda_1+\lambda_2)\rho)c_v V(y_c) \\ & + \frac{c_v\varrho_1}{\kappa}\exp(\ln(\mu)\tau_1 + (\lambda_1+\lambda_2)\tau_2) \\ & + \min_{\hat{q}\in SGN(\zeta_v)}\max_{\varsigma\in SGN(\zeta_v)}\left(\widetilde{w}^T M^T\varsigma - \widetilde{w}^T M^T\hat{q}\right). \end{aligned} \tag{7.24}$$

In (7.24), note that $(\alpha(k_c + \kappa) + I_N) \otimes I_n - Q\kappa_I = 0$.

Define $\lambda := \lambda_1 - \frac{\delta\ln(\mu)}{\kappa} - \rho(\lambda_1 + \lambda_2)$. According to (7.19) and Assumption 7.5, we have

$$\begin{aligned} \min_{\upsilon\in\partial U(\zeta)}\max_{q\in F(\zeta)} \upsilon^T q \le{}& -\zeta_v^T\eta\zeta_v - \alpha\widetilde{w}^T\widetilde{w} - (|M\kappa_p| - c_1|Q\kappa_p| \\ & - \alpha|\beta_I| - \alpha c_1 - |Q|c_2)|\zeta_v| - (k - b_k)v^T v + \widetilde{w}^T v \\ & - v^T G^{\nabla}(\chi + x^*, r\vartheta_1 + R\vartheta_2 + s^*) + b_k\widetilde{w}^T\chi \\ & - b_k\chi^T G^{\nabla}(\chi + x^*, r\vartheta_1 + R\vartheta_2 + s^*) + \alpha\ell_g|\zeta_v||v| \\ & + \ell_g|Q||\widetilde{w}||v| - \lambda c_v V(y_c) + \frac{c_v\varrho_1}{\kappa}\gamma_4, \end{aligned} \tag{7.25}$$

where $\ell_g := \sqrt{\ell_{g_1}^2 + \cdots + \ell_{g_N}^2}$ with $\ell_{g_i} := \left|\frac{\partial g_i(\chi + x^*)}{\partial(\chi + x^*)}\right|$, $\Gamma := \exp(\ln(\mu)N_0 + (\lambda_1 + \lambda_2)T_0)$, $\eta := diag\{\eta_i\}$ satisfying $0 < \eta_i \le (k_i + \kappa_i) - \alpha(k_i\kappa_i),\ \forall i \in \mathscr{V}$.

According to (7.2), Lemma 7.1, (7.10) and (7.14), we know $\Delta^* := G^{\nabla}(\chi^* + x^*, r\vartheta_1^* + R\vartheta_2^* + \frac{1}{N}\sum_{i=1}^{N}\varphi_i(x_i^*)) = \mathbf{0}$. Denote $\Delta_1 := G^{\nabla}(\chi + x^*, r\vartheta_1 + R\vartheta_2 + s^*)$, $\Delta_2 := G^{\nabla}(\chi + x^*, \mathbf{1}_N \otimes \sigma(\chi + x^*))$. Then it holds

$$
\begin{aligned}
-\chi^T\Delta_1 =& -\chi^T(\Delta_1 - \Delta_2 + \Delta_2 - \Delta^*)\\
=& -\chi^T(F_{\nabla}(\chi + x^*) - F_{\nabla}(\chi^* + x^*)) - \chi^T(\Delta_1 - \Delta_2),\\
-v^T\Delta_1 =& -v^T(F_{\nabla}(\chi + x^*) - F_{\nabla}(\chi^* + x^*)) - v^T(\Delta_1 - \Delta_2).
\end{aligned}
\tag{7.26}
$$

Moreover, from Assumption 7.4 we conclude that for any $\bar{r} > 0$, there exist constants $\ell^1_{(x^*,\bar{r})}, \ell^2_{(x^*,\bar{r})}, \bar{\ell}_g > 0$ such that for all $\zeta \in \mathbb{B}_{\bar{r}}$

$$
\begin{aligned}
-\chi^T(\Delta_1 - \Delta_2) &\le \ell^1_{(x^*,\bar{r})}|\chi||r\vartheta_1 + R\vartheta_2|,\\
-v^T(\Delta_1 - \Delta_2) &\le \ell^1_{(x^*,\bar{r})}|v||r\vartheta_1 + R\vartheta_2|,\\
-v^T(\Delta_2 - \Delta^*) &\le \ell^2_{(x^*,\bar{r})}|v||\chi|,\ \ell_g \le \bar{\ell}_g.
\end{aligned}
\tag{7.27}
$$

Furthermore, it follows from (7.26), (7.27) and Assumption 7.2 that

$$
\begin{aligned}
-b_k\chi^T\Delta_1 \le& -b_k m_f \chi^T\chi + b_k\ell_1|\chi||r\vartheta_1 + R\vartheta_2|,\\
-v^T\Delta_1 \le& \ell_2|v||\chi| + \ell_1|v||r\vartheta_1 + R\vartheta_2|,
\end{aligned}
\tag{7.28}
$$

where $\ell_1 := \ell^1_{(x^*,\bar{r})}$ and $\ell_2 := \ell^2_{(x^*,\bar{r})}$.

Applying Young's inequality, we have

$$
\begin{aligned}
& b_k\ell_1|\chi||r\vartheta_1 + R\vartheta_2| \le \frac{b_k m_f}{4}\chi^T\chi + \frac{b_k\ell_1^2}{m_f}(\vartheta_1^T\vartheta_1 + \vartheta_2^T\vartheta_2),\\
& \alpha\bar{\ell}_g|\zeta_v||v| \le \frac{1}{2}\zeta_v^T\eta\zeta_v + \frac{\alpha^2\bar{\ell}_g^2}{2}v^T\eta^{-1}v,\ \widetilde{w}^Tv \le \frac{k}{4}v^Tv + \frac{1}{k}\widetilde{w}^T\widetilde{w},\\
& \ell_1|v||r\vartheta_1 + R\vartheta_2| \le \frac{a_f}{4}(\vartheta_1^T\vartheta_1 + \vartheta_2^T\vartheta_2) + \frac{\ell_1^2}{a_f}v^Tv,\\
& \ell_2|v||\chi| \le \frac{b_k m_f}{4}\chi^T\chi + \frac{\ell_2^2}{b_k m_f}v^Tv,
\end{aligned}
\tag{7.29}
$$

where constants $b_k, a_f > 0$ can be arbitrarily chosen.

Then according to (7.25)–(7.29), we have for all $\zeta \in \mathbb{B}_{\bar{r}}$

$$
\begin{aligned}
\min_{v\in\partial U(\zeta)}\max_{q\in F(\zeta)} v^Tq \le& -\frac{1}{2}\zeta_v^T\eta\zeta_v - \left(\frac{\alpha}{2} - \frac{1}{k}\right)\widetilde{w}^T\widetilde{w} - \frac{b_k m_f}{4}\chi^T\chi\\
& - \left(\frac{k}{6} - \frac{\ell_2^2}{b_k m_f} - \frac{\ell_1^2}{a_f} - b_k\right)v^Tv + \left(\frac{a_f}{4} + \frac{b_k\ell_1^2}{m_f}\right)(\vartheta_1^T\vartheta_1 + \vartheta_2^T\vartheta_2)\\
& - \frac{b_k m_f}{4}\chi^T\chi + b_k\widetilde{w}^T\chi - \frac{\alpha}{4}\widetilde{w}^T\widetilde{w}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\alpha}{4}\widetilde{w}^T\widetilde{w}+\bar{\ell}_g|Q||\widetilde{w}||v|-\frac{2k}{8}v^Tv+\frac{\alpha^2\bar{\ell}_g^2}{2}v^T\eta^{-1}v\\
&-(|M\kappa_p|-c_1|Q\kappa_p|-\alpha|\beta_I|-\alpha c_1-|Q|c_2)|\zeta_v|\\
&-\lambda c_v V(y_c)+\frac{c_v\varrho_1}{\kappa}\Gamma-(\frac{k}{3}-\frac{\ell_2^2}{b_k m_f}-\frac{\ell_1^2}{a_f}-b_k)v^Tv. \quad (7.30)
\end{aligned}
$$

For fixed $k_i, \kappa_i, b, \eta_i, m_f, \bar{\ell}_g, \ell_1, \ell_2, a_f$ with $b > 1$, choose $b_k, k^* > 0$ such that for all $k > k^*$ and $0 < b_k < k$

$$\overline{h} < \alpha^2 \leq \underline{h}, \quad \frac{k}{6}-\frac{\ell_2^2}{b_k m_f}-\frac{\ell_1^2}{a_f}-b_k > 0, \tag{7.31}$$

with $\underline{h} := \min_{i\in\mathscr{V}}\left\{\frac{k\eta_i}{4\bar{\ell}_g^2}, \frac{(k_i+\kappa_i-\eta_i)^2}{(k_i\kappa_i)^2}, \frac{b}{k_i\kappa_i}, \frac{(b-1)^2}{(k_i+\kappa_i)^2}\right\}$, $\overline{h} := \max_{i\in\mathscr{V}}\left\{\frac{16b_k^2}{m_f^2}, \frac{4}{k^2}, \frac{64\bar{\ell}_g^4 b^4}{k^2(k_i\kappa_i)^4}\right\}$, and then we obtain

$$-\frac{b_k m_f}{4}\chi^T\chi+b_k\widetilde{w}^T\chi-\frac{\alpha}{4}\widetilde{w}^T\widetilde{w}\leq 0, \ \alpha(k_i^c+\kappa_i)+1\leq b,$$

$$-\frac{\alpha}{4}\widetilde{w}^T\widetilde{w}+\bar{\ell}_g|Q||\widetilde{w}||v|-\frac{k}{8}v^Tv\leq 0, \ -\frac{k}{8}v^Tv+\frac{\alpha^2\bar{\ell}_g^2}{2}v^T\eta^{-1}v\leq 0,$$

where Q is from (7.22) and η_i is from (7.25).

For fixed $\alpha, b_k, k_i, \kappa_i, b, \eta_i$, there exists a constant $\beta^* > 0$ such that for all $\beta_i > \beta^*$

$$\Delta_3 := (|M\kappa_p|-c_1|Q\kappa_p|-\alpha|\beta_I|-\alpha c_1-|Q|c_2) > 0, \tag{7.32}$$

where β_I is defined in (7.13).

For fixed a_f, b_k, ℓ_1, m_f, choose c_v to be sufficiently large such that

$$-\frac{\lambda}{2}c_v V(y_c)+(\frac{a_f}{4}+\frac{b_k\ell_1^2}{m_f})(\vartheta_1^T\vartheta_1+\vartheta_2^T\vartheta_2)\leq 0. \tag{7.33}$$

For fixed $a_f, b_k, c_v, \ell_1, \ell_2, m_f, \kappa$, there exists a constant $k^* > 0$ such that for all $k > k^*$ and for all $\zeta \in \mathbb{B}_{\bar{r}}$

$$-\frac{\lambda}{4}c_v V(y_c)+\frac{c_v\varrho_1}{\kappa}\Gamma-(\frac{k}{3}-\frac{\ell_2^2}{2b_k m_f}-\frac{\ell_1^2}{2a_f}-\frac{b_k}{2})v^Tv\leq 0. \tag{7.34}$$

Then, from (7.30)–(7.34) and Lemma 7.10, it yields that for all $\zeta \in \mathbb{B}_{\bar{r}}$

$$\begin{aligned}\langle \nabla U(\zeta), F(\zeta)\rangle &\leq \min_{\upsilon\in\partial U(\zeta)} \max_{q\in F(\zeta)} \upsilon^T q\\ &\leq -\frac{1}{2}\zeta_v^T\eta\zeta_v - (\frac{\alpha}{2}-\frac{1}{k})\widetilde{w}^T\widetilde{w} - \frac{b_k m_f}{4}\chi^T\chi\\ &\quad -(\frac{k}{6}-\frac{\ell_2^2}{2b_k m_f}-\frac{\ell_1^2}{2a_f}-\frac{b_k}{2})v^T v - \Delta_3|\zeta_v| - \frac{\lambda}{4}c_v V(y_c);\end{aligned} \tag{7.35}$$

(ii) if $\widetilde{\sigma} \in \mathscr{Q}$, similarly, for all $\zeta \in \mathbb{B}_{\bar{r}}$ we have

$$\begin{aligned}\langle \nabla U(\zeta), F(\zeta)\rangle \leq &-\frac{1}{2}\zeta_v^T\eta\zeta_v - (\frac{\alpha}{2}-\frac{1}{k})\widetilde{w}^T\widetilde{w} - \frac{b_k m_f}{4}\chi^T\chi\\ &-(\frac{k}{6}-\frac{\ell_2^2}{2b_k m_f}-\frac{\ell_1^2}{2a_f}-\frac{b_k}{2})v^T v - \Delta_3|\zeta_v|\\ &-\frac{\lambda}{4}c_v V(y_c).\end{aligned} \tag{7.36}$$

Moreover, according to (7.21), there exist functions $\alpha_1, \alpha_{2,\bar{r}} \in \mathscr{K}_\infty$ such that

$$\alpha_1(|\zeta|_{\mathbb{W}}) \leq U(\zeta) \leq \alpha_{2,\bar{r}}(|\zeta|_{\mathbb{W}}). \tag{7.37}$$

Thus, it follows from (7.21) and (7.35)–(7.37) that there exists a continuous, positive definite function α_4 such that for all $\zeta \in \mathbb{B}_{\bar{r}}$

$$\langle \nabla U(\zeta), F(\zeta)\rangle \leq -\alpha_4(U(\zeta)). \tag{7.38}$$

From μ defined in Theorem 7.7 we have $V_p(y) \leq \mu V_q(y)$, $p, q \in \mathscr{P}$. Then combining (7.17), for all $\zeta \in D$ and all $\zeta^+ \in G(\nu, \widetilde{\sigma}, \tau_1, \tau_2, y)$, we have for all $\zeta \in \mathbb{B}_{\bar{r}}$

$$\begin{aligned}U(\zeta^+) &= W(\nu^+) + \frac{c_v}{\kappa}V_{\widetilde{\sigma}^+}(y^+)\exp(\ln(\mu)\tau_1^+ + (\lambda_1+\lambda_2)\tau_2^+)\\ &\leq U(\zeta).\end{aligned} \tag{7.39}$$

Combining (7.38) and (7.39), for any initial value $|\zeta(0,0)|_{\mathbb{W}} \leq \alpha_{2,\bar{r}}^{-1} \circ \alpha_1(\bar{r})$, it yields that $U(\zeta(t,j)) \leq \alpha_1(\bar{r})$. Then by (7.37), we have $|\zeta(t,j)|_{\mathbb{W}} \leq \bar{r}$. Therefore, it follows that there exists a class-$\mathscr{KL}$ function $\tilde{\beta}$ such that, for any initial value $|\zeta(0,0)|_{\mathbb{W}} \leq \alpha_{2,\bar{r}}^{-1} \circ \alpha_1(\bar{r})$, we have $|\zeta(t,j)|_{\mathbb{W}} \leq \tilde{\beta}(|\zeta(0,0)|_{\mathbb{W}}, t)$, $\forall (t,j) \in$ *dom* ζ. According to (7.5), we have $t \geq \frac{t+j-N_0}{\delta+1}$. Then it follows that $|\zeta(t,j)|_{\mathbb{W}} \leq \tilde{\beta}(|\zeta(0,0)|_{\mathbb{W}}, \max\{\frac{t+j-N_0}{\delta+1}, 0\})$, $\forall (t,j) \in$ *dom* ζ. Thus, the conclusions of Theorem 7.7 follow. □

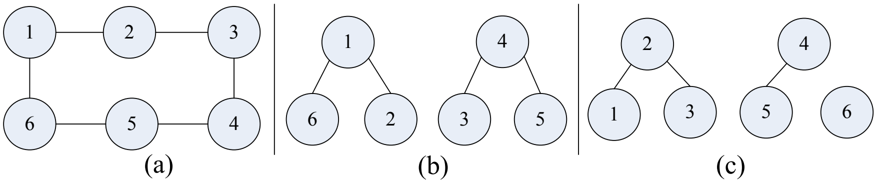


Fig. 7.1 **a** is the original communication topology; **b** and **c** are destroyed communication topologies by two types of attacks

7.6 Example

Here we consider a case with non-quadratic cost functions to verify our algorithm. Consider a six-player network with communication topology described by Fig. 7.1.

The cost functions of these six players, respectively, are

$$\begin{aligned}
J_1(x_1, x_{-1}) &:= x_1^2 + e^{x_1} - q(\sigma(x))x_1 \\
J_2(x_2, x_{-2}) &:= \sin\frac{x_2}{2} + \frac{x_2^2}{2} - q(\sigma(x))x_2 \\
J_3(x_3, x_{-3}) &:= \ln(e^{-0.1x_3} + e^{0.3x_3}) + 0.9x_3^2 - q(\sigma(x))x_3 \\
J_4(x_4, x_{-4}) &:= \frac{x_4^2}{\sqrt{x_4^2+9}} + 0.6x_4^2 - q(\sigma(x))x_4 \\
J_5(x_5, x_{-5}) &:= \cos x_5 + 2x_5^2 - q(\sigma(x))x_5 \\
J_6(x_6, x_{-6}) &:= e^{2x_6} + 3x_6^2 - q(\sigma(x))x_6,
\end{aligned} \tag{7.40}$$

where $q(\sigma(x)) := q_0 - aN\sigma(x)$ with $\sigma(x) := \frac{1}{N}\sum_{i=1}^{N} x_i^2$, and q_0, a are constants.

In the simulation, we first choose $\gamma_0 = 10, \gamma_1 = 2, \gamma_2 = 10, \gamma_3 = 4, \gamma_4 = 25, \kappa = 100$ such that the conclusions of Lemmas 7.4 and 7.5 hold, respectively, and we obtain $\lambda_1 = 0.1007, \lambda_2 = 0.656, \mu = 1.0848$. Secondly, given $q_0 = 200$ and $a = 0.1$, the disturbances $d_i(t) = t, g_i(x) = 2\sum_{j\in\mathcal{N}_i} x_j, i \in \mathcal{V}, k_c = \text{diag}\{1, 3, 4, 2, 3, 4\}, \kappa_c = \text{diag}\{2, 1, 8, 4, 3, 2\}$, $b = 3, \eta_i = 2, b_k = 0.06\sqrt{0.1}$ and $k = 700$ such that (7.31) holds, then we have $\alpha = 0.044$. Thirdly, choose $\beta_c = \text{diag}\{200, 200, 200, 200, 200, 200\}$ such that (7.32) holds, and then choose $c_v = 80.2$ such that (7.33) and (7.34) hold. Finally, correspondingly choose $N_0 = 2, \delta = 0.1, T_0 = 0.5, \rho = 0.12$ such that (7.19) in Theorem 7.7 holds. In addition, from (7.40) we know some gradients of these cost functions are locally, but not globally, Lipschitz continuous. So according to the conclusion in Theorem 7.7, the initial values cannot be allowed to be too far from a NE. Here we choose the initial values of these six players to be $x_1(0) = 2, x_2(0) = 5, x_3(0) = 2, x_4(0) = 3, x_5(0) = 2, x_6(0) = 1$.

One type of switching signal is given in Fig. 7.2. The simulation results are shown in Figs. 7.3–7.5. Figure 7.3 shows that the trajectory of x of the proposed

Fig. 7.2 Switching signal

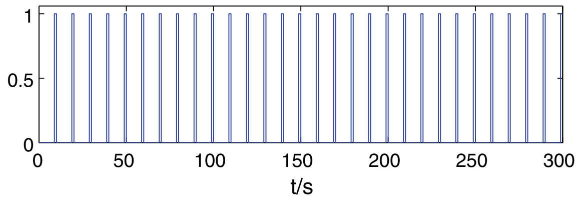
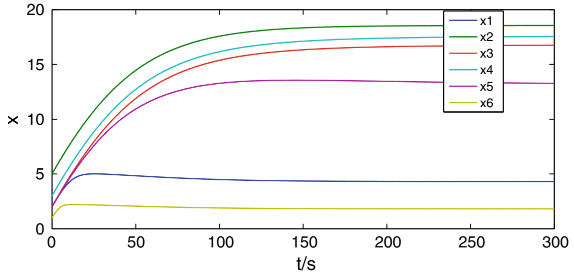


Fig. 7.3 The evolutions of strategy profile under the proposed algorithm



distributed algorithm converges to the NE ($x_1^* = 4.3$, $x_2^* = 18.49$, $x_3^* = 16.69$, $x_4^* = 17.83$, $x_5^* = 13.22$, $x_6^* = 1.81$). Figure 7.4 shows $s_i = 189.25$, $i \in \mathscr{V}$ converge to the aggregative function. Figure 7.5a illustrates that v_i can track the reference signal $\widehat{v}_i$ at steady state. Figure 7.5b illustrates that the observation errors of the players' unmodeled and disturbance terms tend to zero.

Fig. 7.4 The evolutions of s under the proposed algorithm

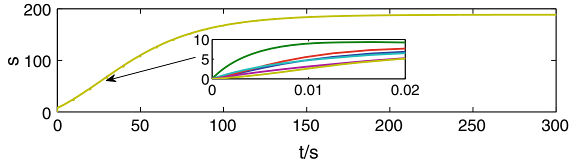
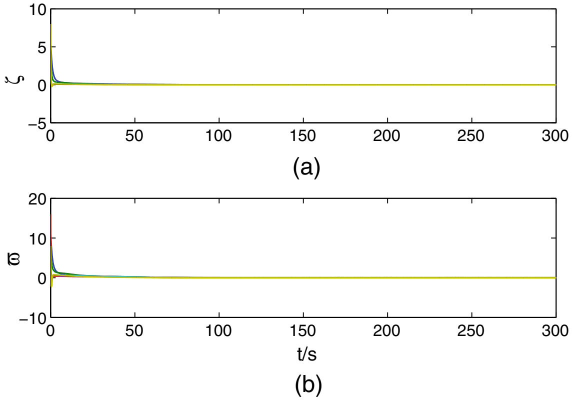


Fig. 7.5 **a** is the evolutions of the errors between the original signal v_i and reference signal $\widehat{v}_i$, $i \in \mathscr{V}$; **b** is the observation errors of the unknown terms



7.7 Conclusions and Future Work

We have studied a robust NE seeking algorithm in aggregative games for double-integrator agents subject to unknown terms. The unknown terms are regarded as an extended-state and an estimator is designed to estimate such a state. In addition, a reference signal is introduced to observe the disturbed state. To obtain the estimation for the aggregate of the decisions of all players, we use a distributed average consensus algorithm by considering the effect of attacks on the communication topology which may be switching and even be unconnected. Moreover, to further analyze the stability of such games, we model the complete closed-loop system as a hybrid system using the ADT automaton and time-ratio monitor to constrain attacks. Then, a Lyapunov function is constructed and an appropriate set corresponding to the NE is shown to be uniformly asymptotically stable. Possible future work is how to achieve a global result, or at least a semi-global result, when we do not have the global Lipschitz assumption on the gradients.

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